VALIDATING THE ANALYTIC HIERARCHY PROCESS AND THE ANALYTIC NETWORK PROCESS WITH APPLICATINS HAVING KNOWN AND MEASURABLE OUTCOMES

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Summary: *Two main ideas are illustrated in this paper: validating the pairwise comparison process and its fundamental scale used in the AHP and the ANP, and validating the network structure used in the ANP. The Saaty compatibility index is used to show the closeness of the derived priorities in the validation examples to actual values against which we wish to compare them that have been standardized to relative form by dividing by their sum. Such examples encourage one about the validity of the AHP and the ANP and its supermatrix as they are applied to real life problems.*

1. Introduction

The story that I want to tell you is about using the human mind as a measuring instrument. We need to be able to validate such measures and the best we can do is to compare our results to things for which we have measurements. They are known as tangibles. We have many examples of such validation of the Analytic Hierarchy Process (AHP) in which the reader can see that indeed the mind is a good measuring instrument. We shall start by showing two validation examples. Then we shall move to validation examples for the Analytic Network Process, estimating market share for companies.

The AHP/ANP is a descriptive psycho-physical theory that aims to use data from the real world, giving back responses to real world decisions in a way that can be compared to the relative values of tangible measurements if they are available, and to propose a new way of doing measurement in a way using judgments that are used to derive relative values (Saaty, 2000). Thus the AHP is a descriptive theory rather than a prescriptive or normative theory. Any truly scientific theory should be descriptive. The physics equation that relates the time it takes a particle to fall from a height is descriptive. In particular, the AHP/ANP can be used as a tool for prediction, a basic requirement in science for a theory to be reliable in terms of cause and effects. The AHP is a practical theory and not top-of-the-head guessing wrapped in a dictum of rationality. Its rationality is shown by how accessible it is to the uninitiated and by the fact that it has been validated over and over again by showing that it produces expected answers.

The fundamental scale of the AHP is: 1 for equal, 3 for moderately more, 5 for strongly more, 7 for very strongly more, and 9 for extremely more with 2, 4, 6, and 8 for intermediate values between. In the fundamental scale, the numbers are absolute numbers meaning how many times more. When a 3 is entered it means 3 times more. One forms the judgment matrix A for a given property. Instead of assigning two numbers w_i and w_j and forming the ratio w_i/w_j we assign a single number $(w_i/w_j)/1$ drawn from the AHP fundamental scale of absolute numbers to represent the dimensionless ratio w_i/w_j , our estimate of the dominance of a_i over a_j with respect to the property. This is an absolute number representing the dominance of a_i over a_j. The relative scale derived from the matrix represents the overall priorities of the alternatives. The derived scale will reveal what w_i and w_j are. By comparing more than two alternatives in a decision problem, one is able to obtain better values for the derived scale because of redundancy in the comparisons, which helps improve the overall accuracy of the judgments. This is a central fact about the relative measurement approach. It needs a fundamental scale to express numerically the relative dominance relationship. A person may not be schooled in the use of numbers but still have feelings and understanding that enable him or her to make accurate comparisons. Such judgments can be applied successfully to compare stimuli that are not too disparate in magnitude. If they are far apart, they are grouped together through a filtering process into clusters each of which includes homogeneous stimuli. By homogeneous we mean fall within specified bounds. The clusters can be appropriately linked through their elements by using a pivot stimulus from a cluster to an adjacent cluster.

2. AHP Validation Examples

In the first of the validation exercises we estimate the relative consumption of beverages showing we can make valid estimates with the AHP based on information in our minds that was gained through prior experience. Perhaps the best of the AHP examples that requires no prior knowledge is the second one of estimating areas. In it we use our eyes and knowledge of what we mean by area to estimate the relative areas of geometric figures.

When the results of a validation exercise are close to numbers that are obtained from some real world problem we have succeeded in validating the AHP in this exercise. If the derived priority vector of absolute numbers in relative form is close to the vector derived by actually measuring the areas and normalizing them, we can start to build up confidence that the AHP process works. To determine the closeness of the AHP results to the actual numbers from real world data use the Saaty compatibility index.

2.1 Estimation of Relative Consumption of Beverages in the United States

Here is an example that shows that the scale works well on homogeneous elements of a real life problem. A matrix of paired comparison judgments is used to estimate relative beverage consumption in the United States. This exercise was done by a group of 30 people who arrived at a consensus for each judgment. The types of beverages are listed on the left and at the top. The judgment is an estimate of how consumption of the drink on the left dominates that of the drink at the top. For example, when the judgment for coffee (row label) versus wine (column label) was made, it was thought that coffee is consumed extremely more and a 9 is entered in the first row and second column position. The value 1/9 is automatically entered in the second row and first column position. If the consumption of a drink on the left does not dominate that of a drink at the top, the reciprocal value is entered. For example in comparing coffee and water in the first row and eighth column position, water is consumed slightly more than coffee and a 1/2 is entered. Correspondingly, a value of 2 is entered in the eighth row and first column position. At the bottom of the Table 1, we see that the derived values obtained by computing the principal eigenvector of the matrix and normalizing it and the actual values obtained from the pages of the Statistical Abstract of the United States are close.

Table 1. Estimating Relative Beverage Consumption.

Which Drink is Consumed More in the U.S.?
An Example of Estimation Using Judgments

Drink Consumption in the U.S.	**Coffee**	**Wine**	**Tea**	**Beer**	**Sodas**	**Milk**	**Water**
Coffee	1	9	3	1	1/2	1	1/2
Wine	1/9	1	1/3	1/9	1/9	1/9	1/9
Tea	1/3	3	1	1/4	1/5	1/4	1/5
Beer	1	9	4	1	1/2	1	1
Sodas	2	9	5	2	1	2	1
Milk	1	9	4	1	1/2	1	1/2
Water	2	9	5	1	1	2	1

The derived scale based on the judgments in the matrix is:

Coffee	Wine	Tea	Beer	Sodas	Milk	Water
.142	.019	.046	.164	.252	.148	.228

The actual consumption for the year 1998 (from Statistical Abstract of the United States, published in 2001) is:

.133	.014	.040	.173	.267	.129	.240

2.2 Estimate the Relative Areas of Geometric Figures

Figure 1 shows five figures. This is a validation exercise for the reader who is invited to apply the paired comparison process of the AHP to determine the relative areas of the figures using his or her own judgments. The object is to obtain the relative weights of the entire group. First one enters one's judgments as to the relative size of each pair of figures in a matrix. In each comparison the smaller figure is used as the unit and one estimates how many times larger the larger one is, using values from the fundamental scale of the AHP. The actual relative values of these areas are A = 0.47, B = 0.05, C = 0.24, D = 0.14, and E = 0.09. If there is no computer available for computing the principal eigenvector of the matrix that contains the derived priorities, a shortcut is to assume it is consistent, normalize each column and then take the average of the corresponding entries in the columns to obtain the priority vector.

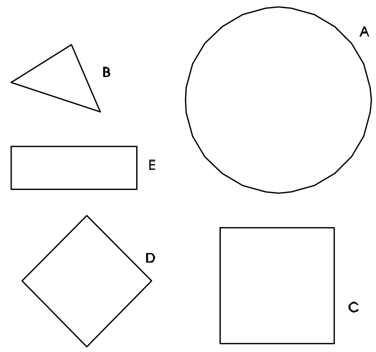


Figure 1. Estimate the Relative Areas of these Figures using AHP.

2.2 The Saaty Compatibility Index

The theory itself provides us with a compatibility index. We denote by $x = (x_i)$, and $y = (y_j)$ respectively the derived and actual scale vectors, and by $c = c_{ij}$ where c_{ij} is obtained as the Hadamard or element-wise product $c_{ij} = (x_i / x_j)(y_j / y_i)$ of one matrix of ratios of the two scales and the transpose of the other matrix of ratios. We then sum the elements of C and divide by n^2 to obtain 1.016 or .016 for deviation from the perfect consistency of the two ratios. This number is much less than the bound 0.1 on inconsistency and incompatibility.

The question remains about the measurement of intangibles. How can one be sure that an expert who is versatile in a certain area including several intangible criteria gives reliable judgments? One of the best examples to validate the interaction of intangible criteria for which we have a method of validation is market share. In it we include no measurements, yet we can check the validity of the process by finding out the dollar share of the companies involved. Even if all criteria were tangible we would still need the AHP/ANP to compare them and trade them off against each other to synthesize a multi-criteria outcome.

I would like to emphasize that multi-criteria measurement is not very useful when one includes irrelevant factors like comparing a stone with an apple according to taste, or forgetting to include relevant and important factors, like other fruits to compare with the apple. That is why including a network of elements tends to correct such deficiencies by including all that one can think of that has bearing on a decision even though no single factor may be the precise one because what we perceive and what is happening out there are separate things.

Our next set of examples involves constructing a network of clusters and elements that influence each other to determine the relative market share of companies using the ANP (Saaty, 2001). We have chosen market share estimations for ANP validation examples because money is a tangible measure that is earned due to a number of intangible physical and behavioral criteria that are intangible that affect the relative standing of a company against its competitors and thus earns it's share of the money in that market. What is impressive here is that the validation includes many criteria that are intangible whose collective interaction leads to a net outcome that can be compared against a tangible known measure such as dollars or number of members. It is hardly possible to steer the results because of the complexity involved and the many judgments that must be made. It has been suggested that to go beyond this example, one could, for example, determine the relative needs of the sectors of a society and validate by comparing the budget allocation of a country. In that case, we would have several criteria similar to market share involved in determining the benefits, opportunities, costs and risks and their final synthesis would be a complicated multi-level decision studied by the ANP.

To create an ANP market estimation model, it helps to be an expert in the topic, because you must draw on your accumulated knowledge. Over the past several years tens of examples of market share models have been done by students with remarkably good results. The subjects have covered such areas as athletic footwear, beverages such as soft drinks and beers, cell phones in Europe, cereals, hotels, household vehicles, internet providers in Korea, health insurance companies in Chile, pizza chains, rental cars, toy retailers, specialty clothing retailers, golf ball manufacturers, telecom companies in Brazil, and world share of energy consumption by continent among others. The results have been surprisingly good in many cases. The ability to get good results seems to depend more on the close familiarity of the person creating the model with the topic than it does on modeling skills. Many of these market share models are included with the sample models in the SuperDecisions software (Creative Decisions Foundation, 2003).

3. ANP Validation Examples -Astounding Results by Only Considering the Influence of Intangibles

The object of this exercise is to try to determine the relative market share of competitors in a particular business, or endeavor, by considering what affects market share in that business and creating a structure of clusters, nodes and influence links. The decision alternatives in such a model are the competitors and the results give their relative dominance. The results can then be compared against some outside measure such as dollars. If dollar income is the measure being used, the incomes of the competitors must be normalized to get their relative share for comparing with the model results.

An ANP model consists of a network of clusters and nodes. Determine the clusters that you think capture the essence of the business. These might be such things as customers, service, economics, advertising, and quality of goods. Then determine what nodes belong in the clusters and enter them. For example, the customers cluster might include nodes for the age groups of the people that buy from the business: teenagers, 20-33 year olds, 34-55 year olds, 55-70 year olds, and over 70. Examine each node in turn and link it to the other nodes in the model that influence it. These nodes will then be pairwise compared with respect to the first node as a "parent" node. When a node in a cluster is linked to nodes in another cluster, an arrow automatically appears going from the first cluster to the second. When a node is linked to nodes in its own cluster, the arrow becomes a loop on that cluster.

The linked nodes in a given cluster are pairwise compared for their influence on the node they are linked from (the parent node) to determine the priority of their influence on the parent node. Comparisons are made as to which is more important to the parent node in capturing "market share". These priorities are then entered in the supermatrix for the network.

The clusters are also pairwise compared to establish their importance with respect to each cluster they are linked from, and the resulting matrix of numbers is used to weight the components of the supermatrix to give the weighted supermatrix. This is then raised to powers until it converges. The synthesized results give the relative market share of the competitors.

If comparison data in terms of sales in dollars, or number of members, or some other known measure are available, one can then construct a compatibility index showing how close the ANP estimated result is to the known measure.

We will give three examples of market share estimation showing details of the process in the first example and showing only the model and its results in the second and third examples.

2.1 Estimating the Relative Market Share of Mass Merchandisers Walmart, Kmart and Target

The ANP model for estimating the relative market share of the mass merchandisers Walmart, KMart and Target using the SuperDecisions software is shown below in Figure 2.

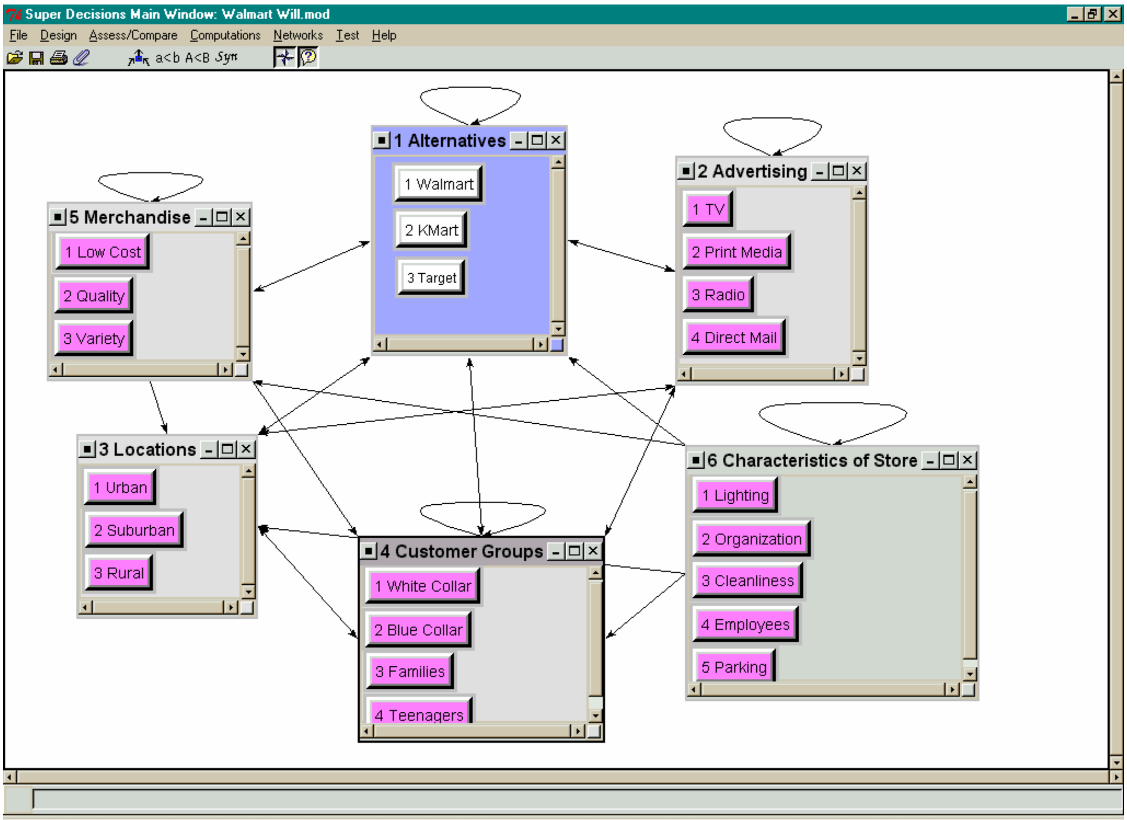


Figure 2. The ANP Model for Estimating the Relative Market Share of Walmart, Kmart and Target.

An ANP model consists of clusters containing elements. The clusters in this case include the alternatives, advertising, locations, customer groups, characteristics of store and merchandise. The elements in the clusters are Walmart, KMart and Target in the Alternatives cluster, Urban, Suburban and Rural in the Locations cluster, and so on as you can see in Figure 2.

Clusters are linked by an arrow whenever one element in the first cluster is linked to several elements in a second cluster. This forms the familiar pairwise comparison group of the AHP with the elements in the second cluster being compared with respect to the element in the first cluster for dominance with respect to it or influence on it. When at least one such link exists between clusters they are shown connected by an arrow in the software. This is termed *outer dependence.* The parent element may be linked to elements in more than one cluster and even to elements in its own cluster. When this occurs it is termed *inner dependence*. Each element in a cluster has the possibility of being a parent, and of having "children" in several other clusters. For each of the pairwise comparison groups judgments are made and the priority vector determined. All such priority vectors are organized in the unweighted supermatrix of the network. The *unweighted supermatrix* is transformed by the matrix of cluster priorities into a column stochastic matrix called the *weighted supermatrix.*

Remark: Raising this matrix to powers represent interactions along paths of length equal to the power of the matrix. It is known that the maximum eigenvalue of a matrix lies between its largest and smallest column sum. The sum of each column of the weighted supermatrix is equal to one and thus its largest eigenvalue is equal to one. By referring to the representation of functions of a matrix (here its powers) in terms of its spectral decomposition in terms of powers of its eigenvalues, we learn that powers of the matrix only converge if the modulus of its largest eigenvalue is equal to one. Because all other eigenvalues are smaller than the largest one which is equal to one, their powers converge to zero. Thus the only chance for a finite set of priorities is when the principal eigenvalue is either equal to one, or is a complex root of one. By Cesaro summability to capture the average sum of interactions for different powers of the weighted supermatrix is the same as taking the limiting powers of the matrix itself. That is what we do here.

The Unweighted Supermatrix

The unweighted supermatrix is constructed of the priorities derived from the pairwise comparison groups. The elements, grouped by the clusters they belong to, are the labels of the rows and columns of the supermatrix. The column for a node *a* contains the priorities of the nodes that have been pairwise compared with respect to *a* in its role as a parent element. The model's supermatrix is shown in Table 2.

Table 2. The Unweighted Supermatrix, Displayed in Two Parts.

Part 1.

Super Decisions Main Window: Walmart Will.mod: Unweighted Super Matrix

	1 Walma~	2 KMart	3 Target	1 TV	2 Print~	3 Radio	4 Direc~	1 Urban	2 Subur~	3 Rural
1 Walma~	0.00000	0.83333	0.83333	0.68698	0.53962	0.63370	0.66076	0.61441	0.65193	0.68334
2 KMart	0.75000	0.00000	0.16667	0.18648	0.29696	0.17436	0.20813	0.26837	0.23506	0.19981
3 Target	0.25000	0.16667	0.00000	0.12654	0.16342	0.19194	0.13111	0.11722	0.11301	0.11685
1 TV	0.55308	0.17595	0.18761	0.00000	0.00000	0.00000	0.00000	0.28750	0.54330	0.55762
2 Print~	0.20192	0.34864	0.42784	0.75000	0.00000	0.80000	0.00000	0.38059	0.23059	0.17540
3 Radio	0.06194	0.05584	0.05483	0.00000	0.00000	0.00000	0.00000	0.05883	0.05290	0.04832
4 Direc~	0.18306	0.41957	0.32972	0.25000	0.00000	0.20000	0.00000	0.27308	0.17321	0.21867
1 Urban	0.11397	0.08362	0.08642	0.44332	0.12601	0.07950	0.09888	0.00000	0.00000	0.00000
2 Subur~	0.40538	0.44429	0.62820	0.38748	0.41612	0.60925	0.53683	0.00000	0.00000	0.00000
3 Rural	0.48064	0.47209	0.28538	0.16920	0.45787	0.31125	0.36429	0.00000	0.00000	0.00000
1 White~	0.14129	0.11364	0.20797	0.16545	0.15543	0.11595	0.12003	0.07842	0.19809	0.09226
2 Blue ~	0.21703	0.21422	0.11737	0.16545	0.15543	0.19809	0.20336	0.22300	0.11595	0.22426
3 Famil~	0.57854	0.62335	0.61954	0.62084	0.64615	0.64097	0.63537	0.65583	0.64097	0.64517
4 Teena~	0.06314	0.04879	0.05512	0.04826	0.04299	0.04499	0.04124	0.04275	0.04499	0.03832
1 Low C~	0.36217	0.33252	0.16766	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2 Quali~	0.26118	0.13965	0.48361	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3 Varie~	0.37665	0.52784	0.34874	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1 Light~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2 Organ~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3 Clean~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4 Emplo~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
5 Parki~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Part II.

Super Decisions Main Window: Walmart Will.mod: Unweighted Super Matrix

1 White~	2 Blue ~	3 Famil~	4 Teena~	1 Low C~	2 Quali~	3 Varie~	1 Light~	2 Organ~	3 Clean~	4 Emplo~	5 Parki~
0.63699	0.66076	0.63010	0.69083	0.66076	0.61441	0.64833	0.66667	0.65481	0.56954	0.64422	0.55842
0.10473	0.20813	0.21844	0.14883	0.20813	0.11722	0.12202	0.11111	0.09534	0.09739	0.08522	0.12196
0.25828	0.13111	0.15146	0.16034	0.13111	0.26837	0.22965	0.22222	0.24986	0.33307	0.27056	0.31962
0.32261	0.51009	0.50836	0.63379	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.21360	0.22102	0.26956	0.17039	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.05936	0.06317	0.04941	0.09609	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.40442	0.20572	0.17266	0.09973	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.16667	0.09362	0.09585	0.10945	0.26837	0.10480	0.09362	0.10050	0.09091	0.09091	0.11111	0.06680
0.83333	0.27969	0.30848	0.30900	0.11722	0.60456	0.62670	0.43306	0.45454	0.45454	0.44444	0.29256
0.00000	0.62670	0.59567	0.58155	0.61441	0.29064	0.27969	0.46644	0.45454	0.45454	0.44444	0.64064
0.00000	0.00000	0.27895	0.08522	0.05089	0.22228	0.16545	0.38337	0.18704	0.24210	0.16545	0.00000
0.00000	0.00000	0.64912	0.17728	0.11226	0.15874	0.16545	0.38337	0.18704	0.20780	0.16545	0.00000
0.85714	0.85714	0.00000	0.73750	0.61767	0.56644	0.62084	0.18510	0.58306	0.49387	0.62084	0.00000
0.14286	0.14286	0.07193	0.00000	0.21919	0.05254	0.04826	0.04816	0.04287	0.05623	0.04826	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.80000	0.80000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.75000	0.00000	0.20000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.25000	0.20000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.16884	0.12074	0.00000	0.25000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.25148	0.00000	0.57500	0.20000	0.75000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.67339	0.46863	0.00000	0.80000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.30787	0.30425	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.07513	0.05466	0.00000	0.00000	0.00000

The Cluster Matrix

The cluster matrix is used to weight the components of the unweighted supermatrix and it is constructed by comparing the importance of the clusters. A component in the supermatrix is all the entries in a cell such as the A cluster impacts another cluster when it is linked from it, that is, when some node in the first cluster is connected to some nodes in the target cluster. A cluster is selected and the clusters linked from it are pairwise compared for importance, resulting in the column of priorities for that cluster. This is done for each cluster in the network and the final result is the cluster matrix shown in Table 3. An interpretation from the numbers in column 1 is that the Merchandise cluster (0.442) and the Locations cluster (0.276) have the most impact on the Alternatives cluster, that is, on Walmart, KMart and Target.

Table 3. The Cluster Matrix used to Weight the Unweighted Supermatrix.

Super Decisions Main Window: Walmart Will.mod: Cluster Matrix View

Cluster Node Labels	1 Alternatives	2 Advertising	3 Locations	4 Customer Groups	5 Merchandise	6 Characteristics of Store
1 Alternatives	0.137180	0.174344	0.093616	0.057188	0.049324	0.037244
2 Advertising	0.091069	0.219895	0.279686	0.234147	0.000000	0.000000
3 Locations	0.276199	0.176285	0.000000	0.168791	0.102082	0.111927
4 Customer Groups	0.053613	0.429476	0.626697	0.539874	0.252118	0.440751
5 Merchandise	0.441939	0.000000	0.000000	0.000000	0.596476	0.316204
6 Characteristics of Store	0.000000	0.000000	0.000000	0.000000	0.000000	0.093874

Done

Weighted Supermatrix

The weighted supermatrix shown in Table 4 is obtained by multiplying a number from the cluster matrix times each number in the corresponding component in the unweighted supermatrix. For example the (1Alternatives, 1Alternatives) number, 0.137180, in the cluster matrix in Table 3, is multiplied times each number in the corresponding component in the unweighted supermatrix shown in Table 2 to yield the numbers in the weighted supermatrix shown in Table 4. The weighted supermatrix is stochastic; that is, each column in Table 4 sums to 1.

Super Decisions Main Window: Walmart Will.mod:

	1 Walma~	2 KMart	3 Target
1 Walma~	0.00000	0.83333	0.83333
2 KMart	0.75000	0.00000	0.16667
3 Target	0.25000	0.16667	0.00000

Figure 3. The (Alternatives, Alternatives) Component in the Unweighted Supermatrix.

Super Decisions Main Window: Walmart Will.mod:

	1 Walma~	2 KMart	3 Target
1 Walma~	0.00000	0.11432	0.11432
2 KMart	0.10289	0.00000	0.02286
3 Target	0.03429	0.02286	0.00000

Figure 4. The (Alternatives, Alternatives) Component in the Weighted Supermatrix.

A blownup view of this is shown in Figure 3 and Figure 4. All the values in the component shown in Figure 3 are multiplied by 0.137180, the value in the corresponding (1,1) position in the cluster matrix, to yield the values in Figure 4.

Table 4. The Weighted Supermatrix Displayed in Two Parts.

Part I.

Super Decisions Main Window: Walmart Will.mod: Weighted Super Matrix

	1 Walma~	2 KMart	3 Target	1 TV	2 Print~	3 Radio	4 Direc~	1 Urban	2 Subur~	3 Rural
1 Walma~	0.00000	0.11432	0.11432	0.11977	0.12060	0.11048	0.14767	0.05752	0.06103	0.06397
2 KMart	0.10289	0.00000	0.02286	0.03251	0.06637	0.03040	0.04651	0.02512	0.02201	0.01870
3 Target	0.03429	0.02286	0.00000	0.02206	0.03652	0.03346	0.02930	0.01097	0.01058	0.01094
1 TV	0.05037	0.01602	0.01709	0.00000	0.00000	0.00000	0.00000	0.08041	0.15195	0.15596
2 Print~	0.01839	0.03175	0.03896	0.16492	0.00000	0.17592	0.00000	0.10645	0.06449	0.04906
3 Radio	0.00564	0.00509	0.00499	0.00000	0.00000	0.00000	0.00000	0.01645	0.01480	0.01351
4 Direc~	0.01667	0.03821	0.03003	0.05497	0.00000	0.04398	0.00000	0.07638	0.04844	0.06116
1 Urban	0.03148	0.02310	0.02387	0.07815	0.02848	0.01402	0.02235	0.00000	0.00000	0.00000
2 Subur~	0.11197	0.12271	0.17351	0.06831	0.09403	0.10740	0.12131	0.00000	0.00000	0.00000
3 Rural	0.13275	0.13039	0.07882	0.02983	0.10347	0.05487	0.08232	0.00000	0.00000	0.00000
1 White~	0.00758	0.00609	0.01115	0.07106	0.08557	0.04980	0.06608	0.04915	0.12414	0.05782
2 Blue ~	0.01164	0.01149	0.00629	0.07106	0.08557	0.08507	0.11196	0.13975	0.07266	0.14054
3 Famil~	0.03102	0.03342	0.03322	0.26664	0.35573	0.27528	0.34980	0.41101	0.40169	0.40432
4 Teena~	0.00339	0.00262	0.00296	0.02073	0.02367	0.01932	0.02270	0.02679	0.02820	0.02401
1 Low C~	0.16006	0.14695	0.07409	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2 Quali~	0.11542	0.06172	0.21372	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3 Varie~	0.16646	0.23327	0.15412	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1 Light~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2 Organ~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3 Clean~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4 Emplo~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
5 Parki~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Part II.

Super Decisions Main Window: Walmart Will.mod: Weighted Super Matrix

1 White~	2 Blue ~	3 Famil~	4 Teena~	1 Low C~	2 Quali~	3 Varie~	1 Light~	2 Organ~	3 Clean~	4 Emplo~	5 Parki~
0.03643	0.03779	0.03603	0.03951	0.03259	0.03030	0.03198	0.03631	0.02439	0.03102	0.03509	0.08557
0.00599	0.01190	0.01249	0.00851	0.01027	0.00578	0.00602	0.00605	0.00355	0.00530	0.00464	0.01869
0.01477	0.00750	0.00866	0.00917	0.00647	0.01324	0.01133	0.01210	0.00931	0.01814	0.01474	0.04898
0.07554	0.11944	0.11903	0.14840	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.05001	0.05175	0.06312	0.03990	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.01390	0.01479	0.01157	0.02250	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.09470	0.04817	0.04043	0.02335	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.02813	0.01580	0.01618	0.01847	0.02740	0.01070	0.00956	0.01645	0.01018	0.01488	0.01819	0.03076
0.14066	0.04721	0.05207	0.05216	0.01197	0.06171	0.06397	0.07089	0.05088	0.07440	0.07275	0.13473
0.00000	0.10578	0.10054	0.09816	0.06272	0.02967	0.02855	0.07635	0.05088	0.07440	0.07275	0.29503
0.00000	0.00000	0.15060	0.04601	0.01283	0.05604	0.04171	0.24711	0.08244	0.15605	0.10664	0.00000
0.00000	0.00000	0.35044	0.09571	0.02830	0.04002	0.04171	0.24711	0.08244	0.13394	0.10664	0.00000
0.46275	0.46275	0.00000	0.39816	0.15573	0.14281	0.15653	0.11931	0.25698	0.31833	0.40018	0.00000
0.07712	0.07712	0.03883	0.00000	0.05526	0.01325	0.01217	0.03105	0.01889	0.03624	0.03111	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.47718	0.47718	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.44736	0.00000	0.11930	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.14912	0.11930	0.00000	0.00000	0.31620	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.01585	0.01658	0.00000	0.09656
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.03452	0.00000	0.07894	0.02746	0.28968
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.09244	0.04399	0.00000	0.10983	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02890	0.04177	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.01031	0.00513	0.00000	0.00000	0.00000

Limiting Supermatrix

The limiting supermatrix shown in Table 5 is obtained from the weighted supermatrix by raising it to powers until it converges. All columns in the limiting supermatrix are identical.

Table 5. Limiting Supermatrix

Part I.

Super Decisions Main Window: Walmart Will.mod: Limit Matrix

	1 Walma~	2 KMart	3 Target	1 TV	2 Print~	3 Radio	4 Direc~	1 Urban	2 Subur~	3 Rural
1 Walma~	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696
2 KMart	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356
3 Target	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465
1 TV	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921
2 Print~	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310
3 Radio	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908
4 Direc~	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887
1 Urban	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161
2 Subur~	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206
3 Rural	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861
1 White~	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801
2 Blue ~	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466
3 Famil~	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983
4 Teena~	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556
1 Low C~	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292
2 Quali~	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366
3 Varie~	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765
1 Light~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2 Organ~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3 Clean~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4 Emplo~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
5 Parki~	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Part II.

Super Decisions Main Window: Walmart Will.mod: Limit Matrix

1 White~	2 Blue ~	3 Famil~	4 Teena~	1 Low C~	2 Quali~	3 Varie~	1 Light~	2 Organ~	3 Clean~	4 Emplo~	5 Parki~
0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696	0.05696
0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356	0.02356
0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465	0.01465
0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921	0.07921
0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310	0.05310
0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908	0.00908
0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887	0.03887
0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161	0.02161
0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206	0.06206
0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861	0.06861
0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801	0.06801
0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466	0.12466
0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983	0.23983
0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556	0.03556
0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292	0.04292
0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366	0.03366
0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765	0.02765
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Synthesized Results

The relative market shares of the alternatives, 0.599, 0.248 and 0.154 are displayed as synthesized results in the SuperDecisions software, shown in Figure 5. These results are obtained from the Limiting Supermatrix shown in Table 5. First note that all columns in this supermatrix are identical. The Raw results for Walmart, Kmart and Target are the first three numbers in column 1: 0.05696, 0.02356 and 0.01465. These are the values in the third column in Figure 5 that are labeled Raw. These numbers are normalized to yield the Normalized values shown in the middle column. The Idealized values are obtained from either the Raw or Normalized values by dividing each number in the column by the largest number in that column.

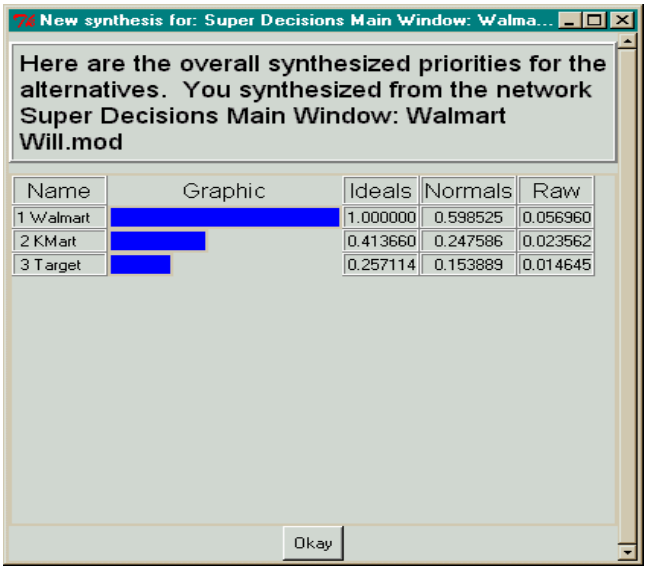


Figure 5. The Synthesized Results Showing the Relative Market Share of the Competitors.

Comparing ANP Results with Actual Relative Market Share of Walmart, KMart and Target

This exercise was to estimate the market share of Walmart, Kmart, and Target. At the time it was done, in early 2001, the latest sales data available were for mid-July, 1998. The sales dollars of the three mass merchandisers, as reported in the Discount Store News, July 13, 1998, p.77, were $58, $27.5 and $20.3 billions of dollars respectively. Normalizing yields actual relative market share values of 54.8, 25.9 and 19.2. The Saaty Compatibility Index was computed and found to be 1.016. This index is used to evaluate the closeness of the *Normals* results shown in Figure 5 from the SuperDecisions software to the normalized actual dollar values. The closer the index to 1, the better the results estimated using ANP.

Competitor	ANP Results	Actual Market Share
Walmart	59.8	54.8
Kmart	24.8	25.9
Target	15.4	19.2
	Saaty Compatibility Index	1.016

The Saaty Compatibility Index for the ANP

The value of the Saaty Compatibility Index for the example above was 1.016. It is computed as follows: construct a reciprocal pairwise matrix from the actual market share and multiply times the transpose of the matrix constructed from the estimated market share using the Hadamard method of matrix multiplication. Using this method one multiplies the elements together that are in the corresponding positions in the matrices, that is, multiply the a_{ij} element in the first matrix A times the b_{ij} element in the second matrix B rather than doing the usual row times column and summing method of matrix multiplication. Add across each row of the resulting matrix to get a vector. Add together the elements of this vector and divide the resulting number by n^2 where n is the order of the matrix, 3 in this case. The closer the resulting number is to 1, the closer the estimated values derived using the ANP to the actual market share. This is easy to see for if the matrix from the ANP market share estimation is identical to the matrix of the market share relative data, then the resulting matrix has all 1's, each row sum is n and as there are n rows this yields $n \times n$ or n^2.

2.2 Estimating Relative Market Share of Eight Airlines

An ANP model to estimate the relative market share of eight American Airlines is shown in Figure 6. The synthesized results from the model are shown in Figure 7 below and the comparison with the actual market share is shown in Table 6.

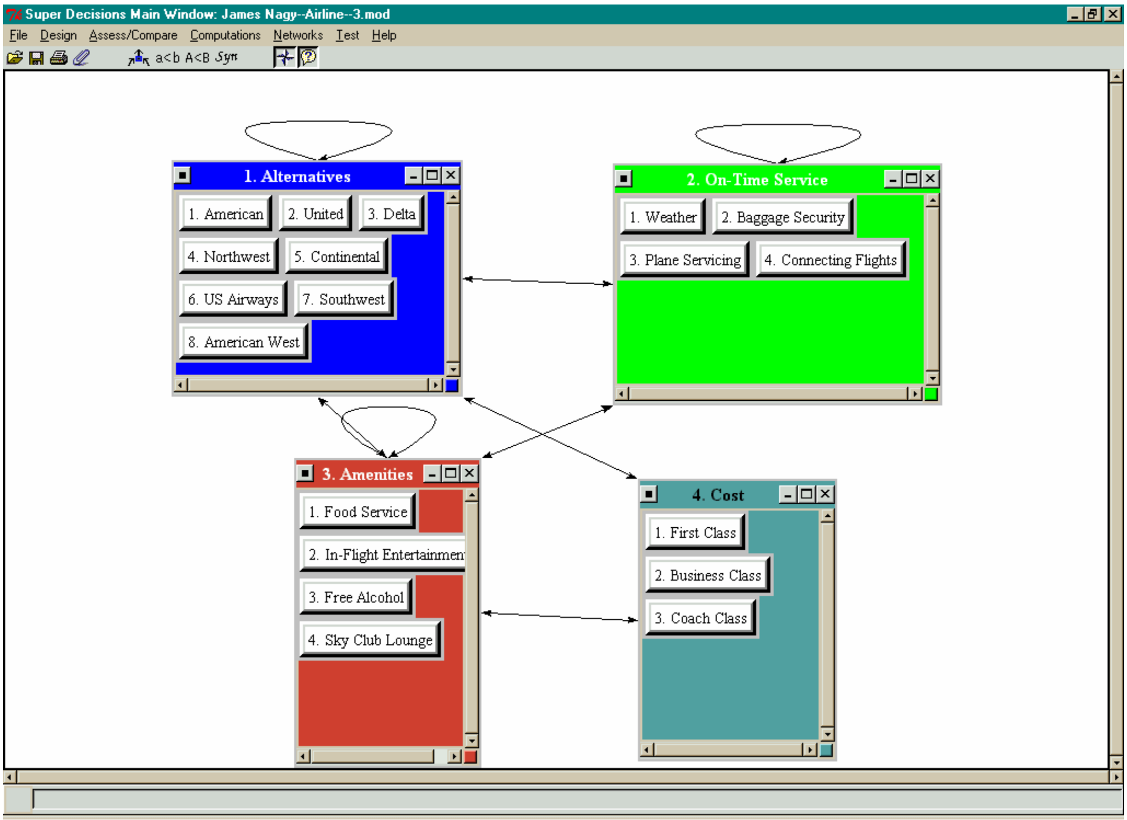
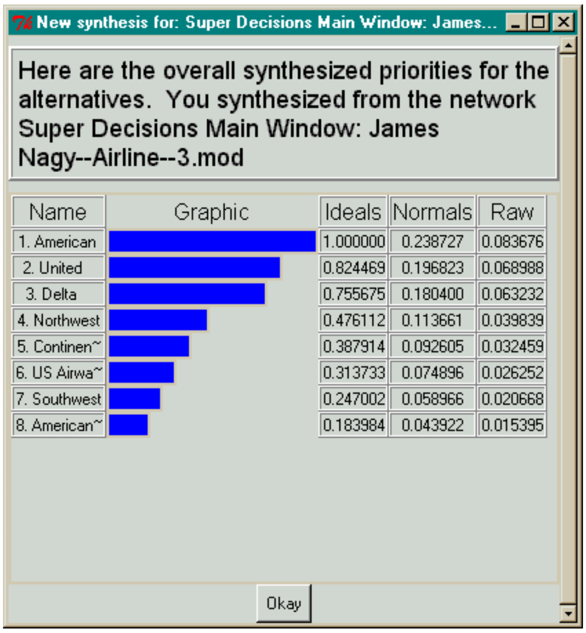


Figure 6. ANP Network to Estimate Relative Market Share of 8 US Airlines.



Name	Graphic	Ideals	Normals	Raw
1. American		1.000000	0.238727	0.083676
2. United		0.824469	0.196823	0.068988
3. Delta		0.755675	0.180400	0.063232
4. Northwest		0.476112	0.113661	0.039839
5. Continen~		0.387914	0.092605	0.032459
6. US Airwa~		0.313733	0.074896	0.026252
7. Southwest		0.247002	0.058966	0.020668
8. American~		0.183984	0.043922	0.015395

Figure 7. Synthesized Results for Airline Market Share

Table 6. Comparing Model Results with Actual Market Share Data

	Model Results	Actual Market Share (yr 2000)
American	23.9	24.0
United	18.7	19.7
Delta	18.0	18.0
Northwest	11.4	12.4
Continental	9.3	10.0
US Airways	7.5	7.1
Southwest	5.9	6.4
American West	4.4	2.9
	Compatibility Index	1.0247

3. Conclusion

In this paper we have shown that the AHP can be validated by constructing a pairwise comparison matrix of judgments on objects that have some physical property for which a scale exists and that we can come close to the relative values of those objects as determined using the scale by expressing judgments in a pairwise comparison matrix and deriving the priority vector.

We have also shown that the ANP can be validated with market share examples using simple network models that include the factors relevant to market share in that area. The results from the model can then be compared to market share as measured by data such as income in dollars.

References

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The SuperDecisions Software (2003), published by Creative Decisions Foundation, available on the internet at www.superdecisions.com.