EVALUATING RELATIONSHIP OF CONSISTENCY RATIO AND NUMBER OF ALTERNATIVES ON RANK REVERSAL

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Summary: *This paper shows a simulation result of rank reversal phenomenon with respect to the changing values of consistency ratio and number of alternatives. The simulation result reveals that number of alternatives and consistency ratio have significant effect to the occurrence of rank reversal. The more the number of alternatives is, the higher the probability of rank reversal occurrence. The more inconsistent a judgment is, the more likely for a rank reversal to occur. This result is based on two-way analysis of variance where the number of alternatives and consistency ratio are considered as factors, and probability of rank reversal occurrence is regarded as response variable. Numbers of alternatives of 4 and 5, and consistency ratio range between 0.02 and 0.10 are chosen to be investigated. Observation is based on rank reversal that occurs after adding a copy of best existing alternative.*

1. Introduction

Rank reversal phenomenon has become a topic of debate for many years. According to Saaty's view, if absolute measurement has been used, there can be no reversal in the rank order of alternatives when new alternatives are added or old ones deleted [Saaty, 1987]. However, if there is no presence of standard, we have to use relative measurement to discover the relative merit of alternatives when compared in paired among themselves. This may lead to the potentiality of rank reversal phenomenon. When someone makes judgment in pairwise comparison, it is probable that he or she becomes inconsistent. AHP technique admits inconsistency in certain level that is less than or equal to 0.1 [Saaty, 1988].

In this paper, a series of simulation is carried out to investigate rank reversal frequency related to the number of alternatives available and the consistency of one's judgment in pairwise comparison assessment. The number of alternatives of 4 and 5, and consistency ratio range from 0.02 until 0.10 are chosen for the purpose of the experiment. The numerical result presented here is for the case of rank reversal phenomenon as a result of adding a new copy of best existing alternative.

2. General Theory

2.1 Rank Reversal

Rank order of the alternatives may change if we add or delete alternative(s) from the existing set of the alternatives. Alternative that is less preferred than another alternative becomes more preferred and vice versa. Here is an example of rank reversal that occurs because of adding a copy of best existing alternative.

Table 1. Before Adding a Copy of Best Existing Alternative

	Alternative 1	Alternative 2	Alternative 3	Alternative 4	Priority	Rank
Alternative1	1	5	4	7	0.6318	1
Alternative2	1/5	1	1	2	0.1461	2
Alternative3	1/4	1	1	1/2	0.1092	3
Alternative4	1/7	½	2	1	0.1129	4

Table 2. After Adding a Copy of Best Existing Alternative

	Alternative 1	Alternative 2	Alternative 3	Alternative 4	Priority	Rank
Alternative1	1	5	4	7	0.3874	1
Alternative2	1/5	1	1	2	0.0870	2
Alternative3	1/4	1	1	1/2	0.0721	4
Alternative4	1/7	1/2	2	1	0.0662	3
Copy of Alternative1	1	5	4	7	0.3874	1

2.2 Reciprocal Matrix

AHP presents the judgments made by decision makers in the form of reciprocal matrix. A reciprocal matrix of comparisons satisfies the property of $X_{ji} = 1/X_{ij}$ for all i, j = 1, 2…, n. Its principal right eigenvector represents priority ordering, which shows decision maker's preference among alternatives. Its largest eigenvalue (l_{max}) represents the measure of consistency.

2.3 Determining Priorities Vector

In order to determine vector of priorities, geometric mean can be used to make rough approximation. The priorities vector is computed by normalizing the result of taking the n^{th} root of multiplied elements of comparisons matrix that lie in a row.

2.4 Consistency Ratio

The AHP deals with consistency explicitly because in making paired comparisons, just as in thinking, people do not have the intrinsic logical ability to always be consistent [Saaty, 1994]. A matrix of comparisons is said to be consistent if its elements satisfy $X_{ij}X_{jk} = X_{ik}$, i, j, k = 1, 2…, n.

Saaty (1988) wrote that AHP estimates consistency by multiplying matrix of comparisons on the right by the vector of priorities to get a new column vector. Then divide first component of new column vector by the first component of priorities vector, the second component of new column vector by the second component of priorities vector, and so on. Later sum over this result and take the average to approximate l_{max}. The closer l_{max} to the number of alternatives is the more consistent the matrix of comparisons. (l_{max} -n)/(n-1), where n is number of alternatives, is called Consistency Index (C.I.). This index shows the amount of deviation from consistency. To determine the goodness of C.I., AHP compares it by Random Index (R.I.), and the result is what we call Consistency Ratio (C.R.). Random Index is the Consistency Index of a randomly generated reciprocal matrix from the scale 1 to 9. The table below shows the value of R.I. sorted by the order of matrix

Table 3. Random Index

Matrix Order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
R.I.	0.00	0.00	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

A matrix of comparisons is considered to be consistent if the value of C.R. is 0.1 or less.

2.5 Two-Way Analysis of Variance (ANOVA)

When there are two or more factors, each at two or more levels, a treatment is defined as a combination of the level of each of the factors [Mitra, 1993]. In order to accommodate possible interaction effect between factors, replication of treatments is employed.

If factor A has *a* levels, factor B has *b* levels, and response variable with *n* times replication is denoted by Y, the data structure of factorial experiment using a completely randomized design can be described as below:

Table 1. Data Structure of Factorial Experiment Using a Completely Randomized Design

	Factor B			
Factor A	1	2	…	*b*
1	$Y_{111}, Y_{112}, \ldots, Y_{11n}$	$Y_{121}, Y_{122}, \ldots, Y_{12n}$	…	$Y_{1b1}, Y_{1b2}, \ldots, Y_{1bn}$
2	$Y_{211}, Y_{212}, \ldots, Y_{21n}$	$Y_{221}, Y_{222}, \ldots, Y_{22n}$	…	$Y_{2b1}, Y_{2b2}, \ldots, Y_{2bn}$
…	…	…	…	…
a	$Y_{a11}, Y_{a12}, \ldots, Y_{a1n}$	$Y_{a21}, Y_{a22}, \ldots, Y_{a2n}$	…	$Y_{ab1}, Y_{ab2}, \ldots, Y_{abn}$

Two-way Analysis of Variance (ANOVA) is employed to observe whether the factors have significant effect towards the response variable.

3. Research Methodology

As the initial phase, *n* order random reciprocal matrices are generated at desired C.R. level. The C.R. level range is from 0.02 to 0.10 and divided into smaller intervals that are: 0.02-0.04, 0.04-0.06, 0.06-0.08 and 0.08-0.10. The random reciprocal matrices contain random variables of the scale 1/9, 1/8, 1/7, ..., 1/2, 1, 2, ...,9, which are equally likely to occur. Then the priorities obtained are sorted to determine the rank order of alternatives. The next step is inserting a copy of best existing alternative to the reciprocal matrix. Again, the priorities are computed. For each random reciprocal matrix, the rank transition is recorded to construct a complete rank transition matrix. The elements of the rank transition matrix represent frequency of rank shifts for all random matrices that are produced. In particular, the rank transition matrix can be expressed as the following:

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{nn} \end{array}$$

i: the rank before a copy of alternative added
j: the rank after a copy of alternative added
a_{ij}: number of rank transition from i to j

After a rank transition matrix is set up, the next computation is done to find the probability of total rank reversal occurrence. Let

$$S_i = \sum_{j=1}^{n} a_{ij} \quad \text{for } i = 1, 2\ldots, n \tag{1}$$

where S_i = sum of transition frequency from i-th rank

Probability of an alternative, which is previously in the i-th rank, moves to the j-th rank after adding of a copy of best alternative will be equal to a_{ij}/S_i and then expressed as p_{ij}. As an example, probability of an alternative, which is previously in the first rank, moves to the second rank after adding of a copy of best alternative will be equal to a_{12}/S_1 and then expressed as p_{12}. A rank transition probability matrix, which sum of its rows equals to 1, is produced as expressed below:

$$\begin{matrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & P_{nn} \end{matrix}$$

i: the rank before a copy of alternative added
j: the rank after a copy of alternative added
p_{ij}: probability of rank transition from i to j

The probability of total rank reversal occurrence can be computed by the following formula.

$$P_{klm} = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}, \text{ where } i \neq j) / (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}) = (\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}, \text{ where } i \neq j) / (\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}) \quad (2)$$

where
P_{klm} = probability of total rank reversal occurrence at k-th row factor, l-th column factor and m-th replication.
k = row factor (number of alternatives) index {1, 2}
l = column factor (C.R. level) index {1, 2, 3, 4}
m = replication index {1, 2, 3}

Two-way Analysis of Variance is employed to test whether there is significant effect of number of alternatives and C.R. interval towards the probability of total rank reversal occurrence (P_{klm}). Combination of number of alternatives and C.R. interval is considered as treatment, whereas the probability of total rank reversal occurrence (P_{klm}) serves as response variable. Table 2 represents the data structure of factorial experiment with 3 replications using a completely randomized design. In this paper, only relatively steady rank reversal probability is used.

Table 2. Data Structure of Factorial Experiment

	Consistency Ratio Interval			
Number of Alternatives	0.02-0.04	0.04-0.06	0.06-0.08	0.08-0.10
4	P_{111} P_{112} P_{113}	P_{121} P_{122} P_{123}	P_{131} P_{132} P_{133}	P_{141} P_{142} P_{143}
5	P_{211} P_{212} P_{213}	P_{221} P_{222} P_{223}	P_{231} P_{232} P_{232}	P_{241} P_{242} P_{243}

4. Data Analysis: An Example

The following illustration will bring forward an example of a rank transition matrix construction. Suppose there are 4 alternatives, thus 4-order random reciprocal matrices are generated. For C.R. interval between 0.06 – 0.08, from 100000 matrices generated, the frequency of rank transition is written in a matrix as follows:

101238	0	0	0
0	97383	3971	23
0	3759	92648	4008
0	20	4227	92723

The matrix above indicates that there are 101238 (a_{11}) first ranks that do not reverse their rank after insertion of a copy of best existing alternative. This condition might take place because there can be two alternatives or more that have the same priority weight. The total sum of the second row equals to 101377 (S_1), which represents the total of second ranks before a copy of best existing alternative is inserted. The second row also describes the frequency of the second rank that moves to the other rank. For instance, a_{23}=3971, this represents the frequency of movement from second to third rank. While the second rank that preserves is a_{22}=97383. The third and fourth rows can be interpreted in the same way.

In order to compute the rank transition probability for each rank, the following steps are employed. First, compute S_i as shown below:

S_1 =	101238
S_2 =	101377
S_3 =	100415
S_4 =	96970

Afterwards, p_{11} is obtained by dividing a_{11} by S_1, p_{12} is obtained by dividing a_{12} by S_1, and p_{21} is obtained by dividing a_{21} by S_2 and so on. Later we can construct the rank transition probability matrix as the following:

1	0	0	0
0	0.9606025	0.0391706	0.0002269
0	0.0374346	0.922651	0.0399144
0	0.0002062	0.0435908	0.9562029

The probability of total rank reversal occurrence (P_{klm}) can be obtained as follows:

$$(a_{12} + a_{13} + a_{14} + a_{21} + a_{23} + a_{24} + a_{31} + a_{32} + a_{34} + a_{41} + a_{42} + a_{43}) / (S_1 + S_2 + S_3 + S_4)$$

or

$$(p_{12} + p_{13} + p_{14} + p_{21} + p_{23} + p_{24} + p_{31} + p_{32} + p_{34} + p_{41} + p_{42} + p_{43}) / 4$$

of which result is equal to 0.04002. After being considered relatively representative, this probability will then be used in Analysis of Variance. Relatively steady condition that supposed to be representative for the rank reversal probability at C.R. interval level of 0.06-0.08 is achieved as shown in the following table and graph:

Table 4. Rank Reversal Probability of 4-Order Random Reciprocal Matrices at C.R. Interval 0.06-0.08

C.R. : 0.06 - 0.08	
No. of Matrices Generated	Probability
2500	0.0414
5000	0.0662
7500	0.0203
10000	0.0089
20000	0.0361
30000	0.0293
40000	0.0329
50000	0.0388
100000	0.0400

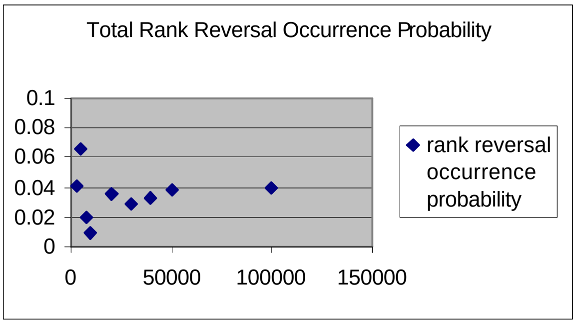

Figure 1. Rank Reversal Probability Graph of 4-Order Random Reciprocal Matrices at C.R. Interval 0.06-0.08

The chart above shows that below 20000 trials of 4-order random reciprocal matrices, the probability is relatively not steady yet. Its deviation still appears to be relatively large. The values of probability lie between 0.01 and 0.07. At 20000 trials of 4-order random reciprocal matrices, rank reversal probabilities range between 0.03 and 0.04. However, if 50000 trials are generated, its probability value appears to be more converged approximately to 0.04. There is no significant difference between the probability obtained from generating 50000 trials of 4-order random reciprocal matrices and the probability obtained from generating 100000 trials of 4-order random reciprocal matrices. Thus, we may consider the probability (P_{klm}) is relatively steady.

Steady state examination of rank transition (p_{ij}) is accomplished in similar way. As an example, here is the rank transition probability of the second rank to the third rank at C.R. interval 0.06-0.08.

Table 5. Rank Transition Probability of 2nd Rank to 3rd Rank of 4-Order Random Reciprocal Matrices at C.R. Interval 0.06-0.08

C.R. : 0.06 - 0.08	
No. of Matrices Generated	Probability
2500	0.0000
5000	0.0481
7500	0.0259
10000	0.0151
20000	0.0553
30000	0.0277
40000	0.0285
50000	0.0430
100000	0.0392

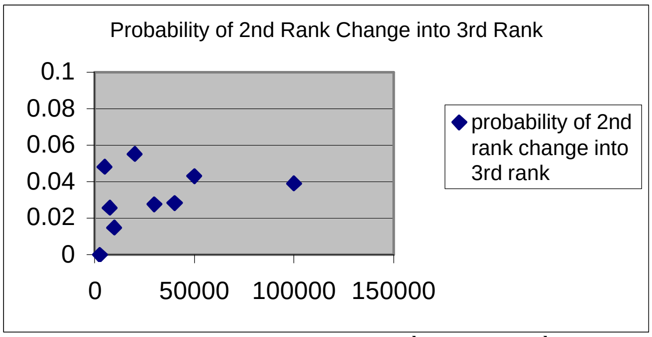

Figure 2. Rank Transition Probability Graph of 2nd Rank to 3rd Rank of 4-Order Random Reciprocal Matrices at C.R. Interval 0.06-0.08

At 50000 trials of matrices, the probability of the second rank will move to the third rank lie between 0.00 and 0.06. After 50000 trials of matrices, the probabilities are around 0.04. There is no significant difference in rank transition probabilities between generating 50000 trials of random matrices and 100000 trials. This condition also happens in another rank transition probability as in Appendix 1.

It is supposed that by generating 100000 trials of random reciprocal matrices, the transition probabilities (p_{ij}) will be more representative. The same procedure will be applied for each replication of the treatment.

Three replications of each treatment are produced for the two-way ANOVA. The treatments that will be used are the combinations of the number of alternatives and C.R. interval. Probabilities of rank reversal occurrence that are obtained will be used as response variable in the Analysis of Variance. The result of two-way analysis of variance is shown below.

Two-way ANOVA: Probability of Rank Reversal Occurrence versus No. of Alternatives, C.R. Interval

```
Analysis of Variance for Probability
Source        DF        SS        MS        F        P
No. of Alter   1 0.0027373 0.0027373   296.41    0.000
C.R. Interval  3 0.0011118 0.0003706    40.13    0.000
Interaction    3 0.0000677 0.0000226     2.44    0.102
Error         16 0.0001478 0.0000092
Total         23 0.0040645

                       Individual 95% CI
No. of Alter    Mean   ----------+---------+---------+---------+-
1            0.03802   (--*--)
2            0.05938                                    (--*--)
                       ----------+---------+---------+---------+-
                           0.04200   0.04800   0.05400   0.06000

                       Individual 95% CI
C.R. Interval   Mean   -------+---------+---------+---------+----
1             0.0406   (----*---)
2             0.0438         (---*---)
3             0.0528                        (---*---)
4             0.0576                                (---*---)
                       -------+---------+---------+---------+----
                         0.0420    0.0480    0.0540    0.0600
```

Figure 3. Two-Way Analysis of Variance

The result shows that p-value of both factors is below 0.05. It substantiates the fact that there is significant effect of number of alternatives and C.R interval towards the probability of rank reversal occurrence. The larger the number of alternatives is the more likely the rank reversal occurrence. The larger the value of the consistency ratio, the more likely the for a rank reversal to occur. From the Analysis of Variance, it is also shown that these data do not provide enough evidence to claim that there is significant interaction effect at α=0.05.

5. Conclusions

It is concluded that there is significant effect of consistency ratio and number of alternatives on the rank reversal probability. The larger the value of consistency ratio, which means judgment made by decision maker is more inconsistent, the higher the potentiality of rank reversal occurrence. The larger the value of the number of alternatives, the higher the potentiality of rank reversal occurrence.

Appendix 1. Rank Transition Probabilities of Random Reciprocal Matrices at C.R. Interval Level of 0.06-0.08

Rank 1	No. of	Probability	Rank 2	No. of	Probability	Rank 3	No. of	Probability	Rank 4	No. of	Probability
change	matrices		change	matrices		change	matrices		change	matrices	
into	2500	1.0000	into	2500	0.0000	into	2500	0.0000	into	2500	0.0000
Rank 1	5000	1.0000	rank 1	5000	0.0000	rank 1	5000	0.0000	rank 1	5000	0.0000
	7500	1.0000		7500	0.0000		7500	0.0000		7500	0.0000
	10000	1.0000		10000	0.0000		10000	0.0000		10000	0.0000
	20000	1.0000		20000	0.0000		20000	0.0000		20000	0.0000
	30000	1.0000		30000	0.0000		30000	0.0000		30000	0.0000
	40000	1.0000		40000	0.0000		40000	0.0000		40000	0.0000
	50000	1.0000		50000	0.0000		50000	0.0000		50000	0.0000
	100000	1.0000		100000	0.0000		100000	0.0000		100000	0.0000
Rank 1	2500	0.0000	Rank 2	2500	1.0000	Rank 3	2500	0.0000	Rank 4	2500	0.0000
change	5000	0.0000	change	5000	0.9520	change	5000	0.0498	change	5000	0.0000
into	7500	0.0000	into	7500	0.9741	into	7500	0.0360	into	7500	0.0000
rank 2	10000	0.0000	rank 2	10000	0.9849	rank 2	10000	0.0123	rank 2	10000	0.0000
	20000	0.0000		20000	0.9447		20000	0.0455		20000	0.0000
	30000	0.0000		30000	0.9722		30000	0.0264		30000	0.0020
	40000	0.0000		40000	0.9715		40000	0.0253		40000	0.0000
	50000	0.0000		50000	0.9551			0.0425		50000	0.0016
	100000	0.0000		100000	0.9606		100000	0.0374		100000	0.0002
Rank 1	2500	0.0000	Rank 2	2500	0.0000	Rank 3	2500	0.9172	Rank 4	2500	0.0828
change	5000	0.0000	change	5000	0.0481	change	5000	0.8676	change	5000	0.0860
into	7500	0.0000	into	7500	0.0259	into	7500	0.9615	into	7500	0.0170
rank 3	10000	0.0000	rank 3	10000	0.0151	rank 3	10000	0.9849	rank 3	10000	0.0054
	20000	0.0000		20000	0.0553		20000	0.9279		20000	0.0161
	30000	0.0000		30000	0.0277		30000	0.9426		30000	0.0302
	40000	0.0000		40000	0.0285		40000	0.9343		40000	0.0376
	50000	0.0000		50000	0.0430		50000	0.9230		50000	0.0319
	100000	0.0000		100000	0.0392		100000	0.9227		100000	0.0436
Rank 1	2500	0.0000	Rank 2	2500	0.0000	Rank 3	2500	0.0828	Rank 4	2500	0.9172
change	5000	0.0000	change	5000	0.0000	change	5000	0.0826	change	5000	0.9140
into	7500	0.0000	into	7500	0.0000	into	7500	0.0024	into	7500	0.9830
rank 4	10000	0.0000	rank 4	10000	0.0000	rank 4	10000	0.0028	rank 4	10000	0.9946
	20000	0.0000		20000	0.0000		20000	0.0266		20000	0.9839
	30000	0.0000		30000	0.0000		30000	0.0310		30000	0.9679
	40000	0.0000		40000	0.0000		40000	0.0404		40000	0.9624
	50000	0.0000		50000	0.0019		50000	0.0345		50000	0.9665
	100000	0.0000		100000	0.0002		100000	0.0399		100000	0.9562

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