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GROUP JUDGMENTS IN FLOOD RISK MANAGEMENT: SENSITIVITY ANALYSIS AND THE AHP

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Summary*:* *Flood management is an important issue in Japan as Japanese rivers are steep in gradient and short in length, and some 120 million people populate the river basin densely. The AHP is shown to be a valuable technology for aggregating group judgments. Sensitivity measures are developed to determine the robustness of the consistency ratio and the principal right eigenvector to perturbations in the group judgments of the pairwise comparison matrix. This paper investigates how uncertainty in each of the input affects the derived output variables.*

1. Introduction

MCDA for flood management is one of the fastest growing areas in operations research, borrowing heavily from fields such as hydrology, geology, psychology and computer science. Most flood management problems in Japan are participatory processes which inherently involve input from a variety of decision makers, whose judgments must be aggregated. In the context of AHP, the aggregation of group judgments in discussed (Section 2). Next, sensitivity metrics are proposed to quantify the notion of robustness to uncertainty. It is shown that the derived consistency ratio (Section 3) and principal eigenvector (Section 4) are insensitive to small perturbation of the consistent pairwise comparison matrix.

2. Aggregation of Individual Judgments and Sensitivity Analysis in AHP

An important problem in decision analysis involves the combination or aggregation of problem-solving knowledge and judgments. In the AHP it is well known that that an *m x m* pairwise comparison matrix requires $\frac{m(m-1)}{2}$ such input judgments. Bolloju (1997) emphasizes the importance of combining or aggregating individual judgments into a group score: “Many typical business environments require such combination or aggregation of decision making for many reasons such as validation, consistency verification, and training. Any technique for elicitation and aggregation of problem-solving knowledge should deal with inconsistencies, conflicts, and decision makers' subjectivity.” In the context of AHP, this paper uses the geometric mean of individual judgments to obtain a combined group judgment. This may help to overcome difficulties arising from a lack of group consensus (Davies, 1994). Other techniques for aggregation include “conceptual aggregation” based on conceptual clustering and case-based learning for real-time (dynamic) decision making (Chaturvedi et al., 1993); the flexible modeling approach based on Bayesian analysis for aggregation of point estimates (Clemen and Winkler, 1993); and aggregation of preference patterns using social choice framework (Dubois and Koning, 1994). Comparative studies on group preference aggregation are reported by Ramanathan and Ganesh (1994) and Perez and Barba-Romero (1995).

2.1 Mean of Individual Judgments

In the Tokai flooding problem, three important criteria are considered: flood frequency (criteria 1), water velocity (criteria 2), and inundation depth (criteria 3). Consider judgment a_{12}, the relative importance of the flood frequency compared with water velocity. The sample geometric mean of a set $\{a_{12}^1, a_{12}^2, a_{12}^3, \ldots, a_{12}^N\}$ of judgments from N experts is defined by:

$$< a_{12} > \; = \; \sqrt[N]{a_{12}^1 \times a_{12}^2 \times a_{12}^3 \times \cdots \times a_{12}^N} \qquad (1)$$

In such a way, a "consensus matrix" can be established by aggregating the individual judgments of the N decision makers using the geometric mean approach. While the "geometric mean method" is the most widely used and theoretically accepted approach in AHP for combining individual judgments to form a group opinion, Ramanathan and Ganesh (1994) showed, using counter-examples, that the geometric mean method does not always satisfy the Pareto optimality axiom, one of the prominent social choice axioms.

2.2 Variance of Individual Judgments

It is also important to consider a measure of data variability. For illustrative purposes, consider again judgment a_{12}, the relative importance of the flood frequency versus water velocity. An unbiased estimator, the sample variance of the judgments, can be established as the mean square deviation of a_{12} values from the geometric mean, $< a_{12} >$:

$$\mathrm{var}[a_{12}] = s^2[a_{12}] = \frac{1}{N-1} \sum_{i=1}^{N} \left(a_{12}^i - < a_{12} > \right)^2 \qquad (2)$$

Since we are eliciting opinions from a sample of *N* decision makers from a possibly large population, the geometric mean $< a_{12} >$ is only an approximation to the "true mean." Because this mean must be estimated from available judgments (data), the number of degrees of freedom is reduced by 1, hence the factor of 1/(*N*-1) in the variance. If we have judgments available from the total population of decision makers, or if the mean was known *a priori*, then the factor would be 1/*N*. Of course the *standard deviation* of judgment a_{12} is well-known to be $s[a_{12}]$, the square root of the variance. The standard deviation gives a measure of the "spread" of the judgments and can also be used to assign probabilities for being within a certain range of values.

Consider now the synthesis of individual judgments for entry a_{12}, assuming that the judgments of three decision makers are as follows:

$$\{a_{12}^1, a_{12}^2, a_{12}^3\} = \{1, 2, 4\} \qquad (3)$$

The combined (group) judgment, using the (geometric) mean should be:

$$< a_{12} > \; = \; \sqrt[3]{1 \times 2 \times 4} = 2 \qquad (4)$$

with corresponding variance:

$$\mathrm{var}(a_{12}) \; = \; \left[\frac{(1-2)^2 + (2-2)^2 + (4-2)^2}{3-1} \right] \; = \; \frac{5}{2} \qquad (5)$$

For convenience, assume that we have the following geometric mean values for each entry in the pairwise comparison matrix for the Tokai flooding problem, giving rise to a perfectly consistent matrix:

$$\begin{bmatrix} 1 & <a_{12}> & <a_{13}> \\ <a_{21}> & 1 & <a_{23}> \\ <a_{31}> & <a_{32}> & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{bmatrix} \quad (6)$$

Similarly, let us assume the following pairwise comparison matrix of variances:

$$\begin{bmatrix} 1 & \mathrm{var}(a_{12}) & \mathrm{var}(a_{13}) \\ \mathrm{var}(a_{21}) & 1 & \mathrm{var}(a_{23}) \\ \mathrm{var}(a_{31}) & \mathrm{var}(a_{32}) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 & 7/2 \\ 2/5 & 1 & 9/2 \\ 2/7 & 2/9 & 1 \end{bmatrix} \quad (7)$$

2.3 Sensitivity Metrics

In this section we develop sensitivity metrics to quantify how output variables are affected by uncertainty in the group judgments of the pairwise comparison matrix. To obtain the eigenvalue and eigenvectors of a 3 by 3 matrix, we must solve the problem

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = l_{\max} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (8)$$

In the cubic and quartic cases, derived output variables, such the consistency ratio, can be explicitly written as a function of the judgments in the pairwise comparison matrix. For example, in the cubic case, Saaty (2003) has explicitly derived $CR = f(a_{12}, a_{13}, a_{23})$ as follows:

$$CR = \frac{CI}{MRCI(n)} = \frac{\sqrt[3]{a_{13}(t)/a_{12}(t)a_{23}(t)} + \sqrt[3]{a_{12}(t)a_{23}(t)/a_{13}(t)} - 2}{1.16} \quad (9)$$

where CR, CI, and MRCI(n) are the Consistency Ratio, Consistency Index and Mean Random Consistency Index. Similarly, the weights, w_i, and the largest eigenvalue, $l_{\max}$, are also a function of the input judgments, hence $w_i = f(a_{12}, a_{13}, a_{23})$ and $l_{\max} = f(a_{12}, a_{13}, a_{23})$. A scenario can be defined as a vector of values for the pairwise comparison matrix input judgments, $\mathbf{a}$, where:

$$\mathbf{a} = (a_{12}, a_{13}, a_{23}) \quad (10)$$

For each input, the geometric mean of the group judgments represent the nominal or "base case" value for each input. Denote these nominal input values, $<a_{12}>$, $<a_{13}>$, and $<a_{23}>$. Together, these three input values specify the nominal scenario:

$$<\mathbf{a}> = (<a_{12}>, <a_{13}>, <a_{23}>) \quad (11)$$

It is of interest to note how an output variable, such as the consistency ratio, changes with the value of the inputs. The corresponding nominal output variable is defined as:

$$< y > = f(< \mathbf{a} >) \tag{12}$$

A sensitivity metric, S_y, is used to quantify the degree to which each of the group input judgments a_{ij} contributes to the uncertainty in the output variable y:

$$S_y(a_{ij}) = \left.\frac{\partial y}{\partial a_{ij}}\right|_{<\mathbf{a}>} \frac{< a_{ij} >}{< y >} \tag{13}$$

Note that all of the partial derivatives are evaluated at the nominal (*i.e.* geometric mean) scenario, $< \mathbf{a} >$. These measures can be combined to determine the *total sensitivity*, S_y^T:

$$S_y^T = S_y(a_{12}) + S_y(a_{13}) + S_y(a_{23}) \tag{15}$$

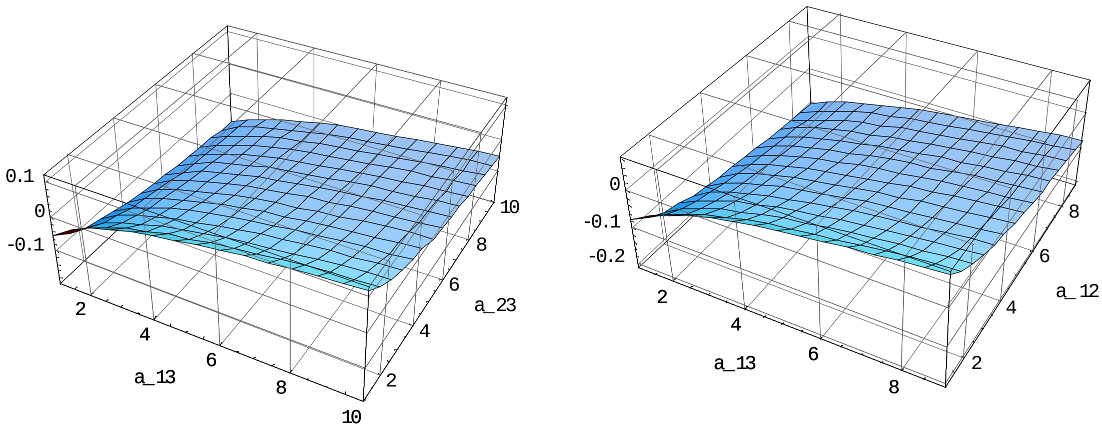
Morgan and Henrion (1990) refer to the metric in Eq. 13 as a measure of 'elasticity', and discuss the use of additional related sensitivity metrics.

3. Sensitivity of the Consistency Ratio to Uncertainty in Group Judgments

This section illustrates that the consistency ratio is robust to uncertainty (small perturbations) in the group judgments of a 3 by 3 pairwise comparison matrix. Using Eq. 13, the sensitivity in the CR arising from uncertainties in judgment a_{12} is denoted:

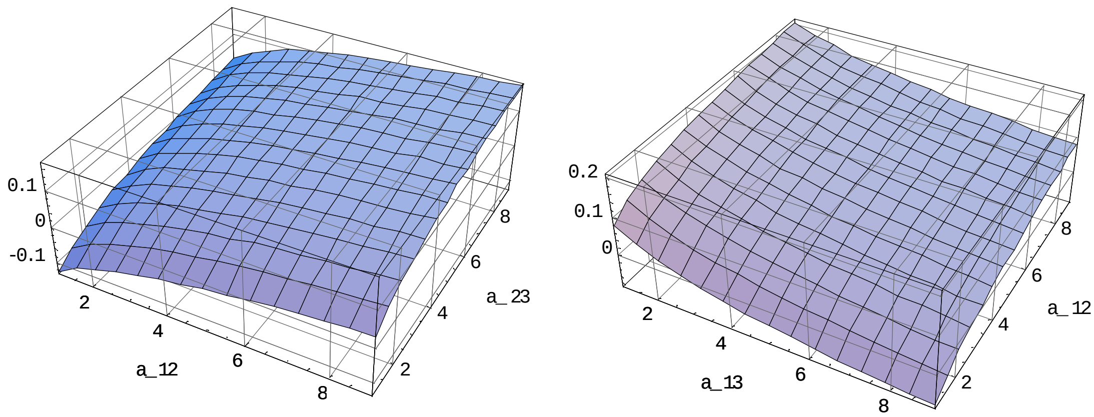
$$S_{CR}(a_{12}) = \left.\frac{\partial y}{\partial a_{12}}\right|_{<\mathbf{a}>} \frac{< a_{12} >}{< y >} \tag{14}$$

The factor $\frac{< a_{12} >}{< y >}$ in Eq (14) ensures that this measure of uncertainty is unaffected by the unit of measurement of both the input and output variables. Similar measures can be defined for the two remaining group judgments a_{13} and a_{23}. As shown in Figures 1 through 3 S_{CR}^T should be relatively small because the individual sensitivities $S_{CR}(a_{13})$, $S_{CR}(a_{12})$, and $S_{CR}(a_{23})$ are shown to be small, when the consistency ratio is the output variable. However, a drawback of this approach is that it does not consider the degree of variation in each input. A group input judgment that has a small sensitivity, but a large variation about its nominal value (due to uncertainty and perhaps disagreements among decision makers in the group) may lead to significant variation in the derived consistency ratio and eigenvector.



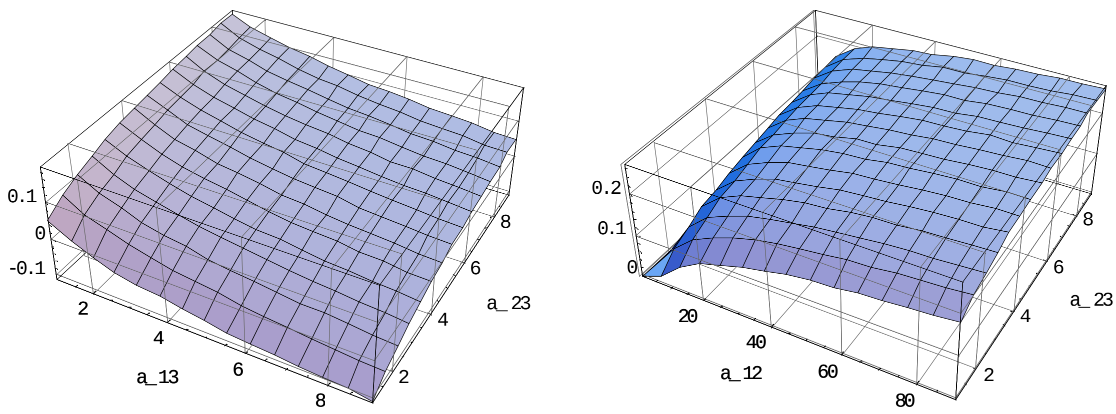
(a) (b)

Figure 1. $S_{CR}(a_{13})$ **as a function of a)** a_{13} and a_{23} **and b)** a_{12} and a_{13}



(a) (b)

Figure 2. $S_{CR}(a_{12})$ **as a function of a)** a_{12} and a_{23} **and b)** a_{12} and a_{13}



(a) (b)

Figure 3. $S_{CR}(a_{23})$ **as a function of a)** a_{13} and a_{23} **and b)** a_{12} and a_{23}

3.1 Gaussian Approximation to measure the variance of the consistency ratio

The Taylor series expansion provides a way to express deviations of an output y from nominal values, $y - \langle y \rangle$, in terms of deviations of inputs from their nominal values: $a_{ij} - \langle a_{ij} \rangle$. Successive terms of the Taylor series contain higher order powers of deviations and higher order derivatives of the function with respect to each input. If the deviations $a_{ij} - \langle a_{ij} \rangle$ are relatively small, then higher powers will become very small. And if the output function is relatively smooth in the region of $\langle \mathbf{a} \rangle$, the higher derivatives will also be small and hence higher order terms can be safely ignored. Under these conditions, the Taylor series produces a good approximation. For example, consider an approximation for the variance of y assuming independence of the a_{12}, a_{13}, and a_{23} input judgments, using only Taylor series terms up to the second order:

$$\mathrm{var}(y) \approx \mathrm{var}(a_{12}) \left.\frac{\partial y}{\partial a_{12}}\right|_{<\mathbf{a}>} \left.\frac{\partial y}{\partial a_{12}}\right|_{<\mathbf{a}>} + \mathrm{var}(a_{13}) \left.\frac{\partial y}{\partial a_{13}}\right|_{<\mathbf{a}>} \left.\frac{\partial y}{\partial a_{13}}\right|_{<\mathbf{a}>} + \mathrm{var}(a_{23}) \left.\frac{\partial y}{\partial a_{23}}\right|_{<\mathbf{a}>} \left.\frac{\partial y}{\partial a_{23}}\right|_{<\mathbf{a}>} \quad (15)$$

From Eq. 15 one can see that if the input judgments are independent, then the variance of the output is approximately the sum of squares of the products of the standard deviation, $s\,(a_{ij})$, and the sensitivity $\frac{\partial y}{\partial a_{ij}}$ of each input:

$$\mathrm{var}(y) \approx \left[s\,(a_{12}) \left.\frac{\partial y}{\partial a_{12}}\right|_{<\mathbf{a}>} \right]^2 + \left[s\,(a_{13}) \left.\frac{\partial y}{\partial a_{13}}\right|_{<\mathbf{a}>} \right]^2 + \left[s\,(a_{23}) \left.\frac{\partial y}{\partial a_{23}}\right|_{<\mathbf{a}>} \right]^2 \quad (16)$$

This implies that the variance of the output is given by a Gaussian approximation, where total uncertainty in the output, expressed as variance, is explicitly decomposed as the sum of uncertainty contributions from the input. We can prove that every pair of input judgments, for example a_{12} and a_{23} are independent by showing that their covariance is zero. The covariance of two elements a_{12} and a_{23} is defined by:

$$\begin{aligned} \mathrm{cov}(a_{12}, a_{23}) &= \langle (a_{12} - m_{12})(a_{23} - m_{23}) \rangle \\ &= \langle a_{12} a_{23} \rangle - \langle a_{12} \rangle \langle a_{23} \rangle \end{aligned} \quad (17)$$

where $m_{12} = \langle a_{12} \rangle$ and $m_{23} = \langle a_{23} \rangle$ are the means of a_{12} and a_{23}, respectively. Now, it is true that for any two distinct judgments in a pairwise comparison matrix, the geometric mean of the product of two judgments is equal to the product of their geometric means. For example:

$$\langle a_{12} a_{23} \rangle = \langle a_{12} \rangle \langle a_{23} \rangle \quad (18)$$

It follows that for a 3x3 matrix $\mathrm{cov}(a_{12}, a_{23}) = 0$, $\mathrm{cov}(a_{23}, a_{13}) = 0$, and $\mathrm{cov}(a_{12}, a_{13}) = 0$. Finally, the covariance of two identical judgments can be expressed as the variance of a single judgment. For example, the covariance between judgment a_{13} and itself becomes:

$$\begin{aligned} \mathrm{cov}(a_{13}, a_{13}) &= \langle a_{13}^2 \rangle - \langle a_{13} \rangle^2 \\ &= \mathrm{var}(a_{13}) \end{aligned} \quad (19)$$

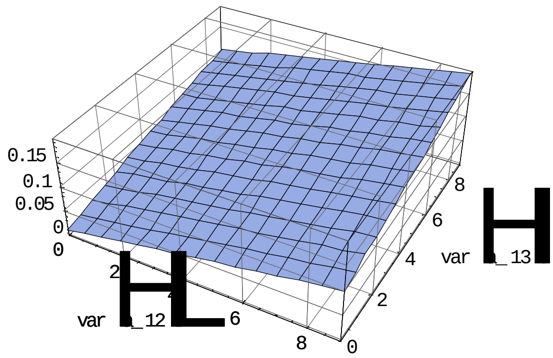


Fig 4. Variance of the Consistency Ratio, $\mathrm{var}(CR)$, as a function of $\mathrm{var}(a_{12})$ and $\mathrm{var}(a_{13})$

The relationship between variations in the inputs, $\mathrm{var}(a_{12})$ and $\mathrm{var}(a_{13})$, and variations in the output, $\mathrm{var}(CR)$, is given in Figure 4 where it is assumed, for illustrative purposes, that $\mathrm{var}(a_{23})=1$ and $S_{CR}(a_{23})=0.1$

4. Sensitivity of Criteria Weights to Perturbations in Input Judgments

Assume that the pairwise comparisons in Figure 5 are obtained by taking the geometric mean and variance of the group judgments. Note that in Figure 5 there are three criteria: flood frequency (criteria A), flood velocity (criteria B), and flood depth (criteria C).

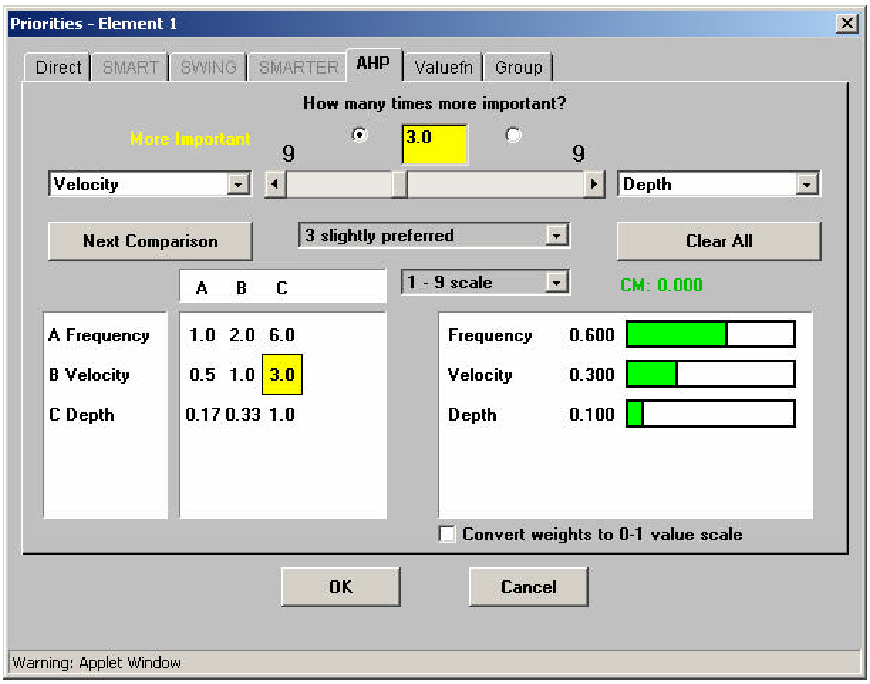


Fig 5. Consistent Matrix for the Flooding Criteria: Frequency, Velocity, and Depth

The weights of criteria A (frequency), B (velocity), and C (depth) are denoted as w_A, w_B, and w_c. Using the sensitivity notation developed earlier, for group input a_{12} the sensitivity in w_A would be:

$$S_{w_A}(a_{12}) = \left.\frac{\partial w_A}{\partial a_{12}}\right|_{\langle \mathbf{a} \rangle} \frac{\langle a_{12} \rangle}{\langle w_A \rangle} \qquad (20)$$

The flooding criteria (w_A) should also be examined for sensitivity to changes in group inputs a_{13} and a_{23}:

$$S_{w_A}(a_{13}) = \left.\frac{\partial w_A}{\partial a_{13}}\right|_{\langle \mathbf{a} \rangle} \frac{\langle a_{13} \rangle}{\langle w_A \rangle} \quad \text{and} \quad S_{w_A}(a_{23}) = \left.\frac{\partial w_A}{\partial a_{23}}\right|_{\langle \mathbf{a} \rangle} \frac{\langle a_{23} \rangle}{\langle w_A \rangle} \qquad (21)$$

We next investigate how uncertainty in the aggregated judgment a_{23} affects the sensitivity of the flooding, velocity, and depth criteria. This is illustrated in Figures 6 through 8 respectively. Note that in Figure 7 the sensitivity metric:

$$S_{w_B}(a_{23}) = \left.\frac{\partial w_B}{\partial a_{23}}\right|_{\langle \mathbf{a} \rangle} \frac{\langle a_{23} \rangle}{\langle w_B \rangle} \qquad (22)$$

is used because we are considering the sensitivity of the velocity criteria (criteria B). Finally, Figure 8 considers the sensitivity of depth criteria (criteria C) to uncertainties in judgment a_{23} and hence uses the metric:

$$S_{w_C}(a_{23}) = \left.\frac{\partial w_C}{\partial a_{23}}\right|_{\langle \mathbf{a} \rangle} \frac{\langle a_{23} \rangle}{\langle w_C \rangle} \qquad (23)$$

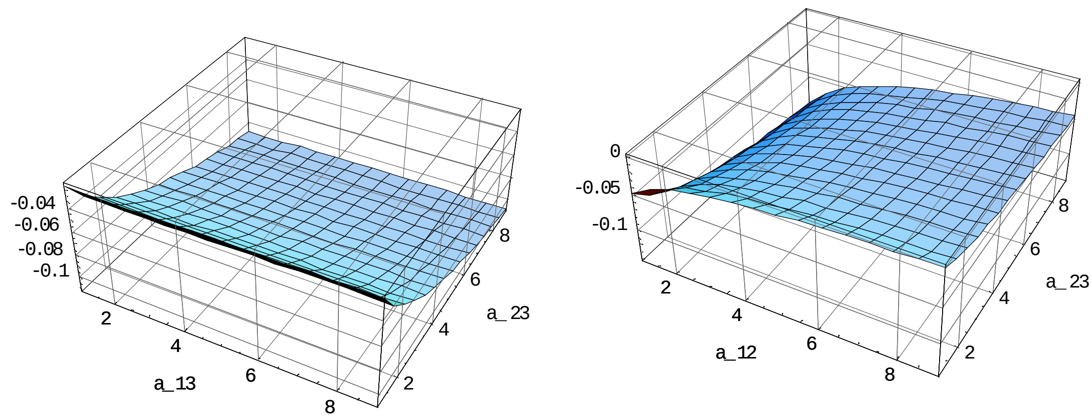


Figure 6. $S_{w_A}(a_{23})$ **as a function of a)** a_{13} and a_{23} **and b)** a_{12} and a_{23}

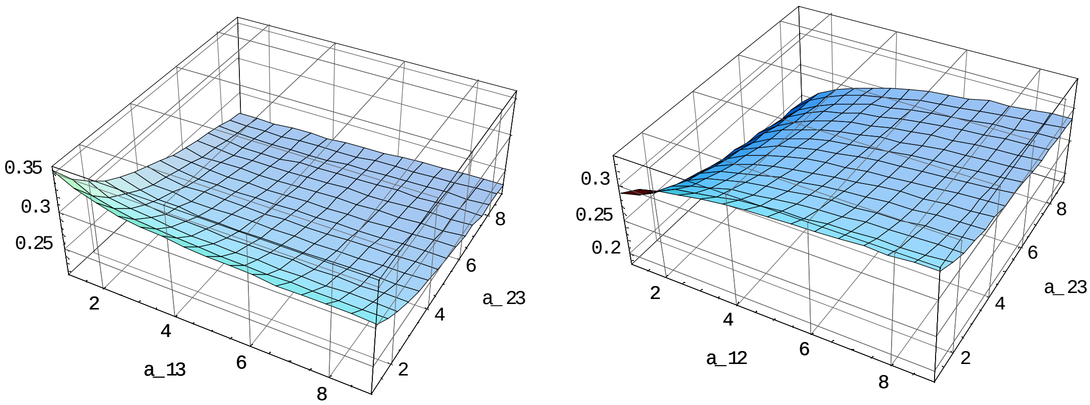


Figure 7. $S_{w_B}(a_{23})$ **as a function of a)** a_{13} and a_{23} **and b)** a_{12} and a_{23}

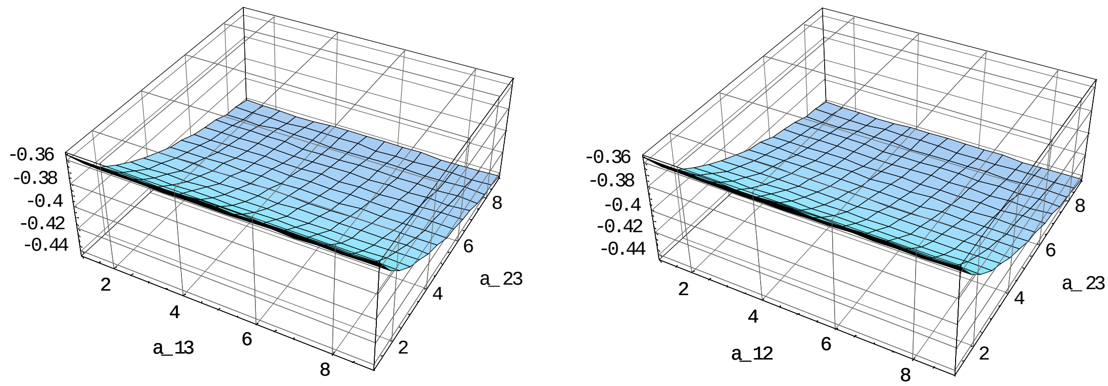


Figure 8. $S_{w_C}(a_{23})$ **as a function of a)** a_{13} **and** a_{23} **and b)** a_{12} **and** a_{23}

Note that:

$$\left|S_{w_C}(a_{23})\right| \succ \left|S_{w_B}(a_{23})\right| \succ \left|S_{w_A}(a_{23})\right| \qquad (24)$$

Hence, we can conclude that the 'weight' of the depth criteria (w_c) is most affected by uncertainties in the group judgment a_{23}. Also, from Eq. 24 we can conclude that the weight of the velocity criteria (w_B) is more sensitive to uncertainties in a_{23} than is the depth criteria (w_c).

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