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COMPLEX NUMBER PAIRWISE COMPARISON AND COMPLEX NUMBER AHP

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Summary: *New mode of ambiguity expression,"complex number", is introduced into pairwise comparison and AHP. The imaginary part of judgment reflects some type of ambiguity. Their applicability is examined through some examples.*

1. Introduction

There can be many kinds of ambiguity modes. Some of them are probability[KITO02], fuzziness[BUCK85], interval expression[ARBE92], and so on. As one of ambiguity expressions, we propose to use "complex number" and to incorporate it into pairwise comparison and AHP, which are called complex number pairwise comparison (C-comparison) and complex number AHP(C-AHP), respectively.

2. Complex number pairwise comparison (C-comparison)

The (j, k)th element of pairwise comparison matrix A, a_{jk}, indicates the dominance of item j over item k, or how many times more important item j is than item k. These a_{jk} data are usually measured in the real scale. Even in case of ambiguity existence, they have been expressed in the forms of probability distribution[KITO02], membership function[BUCK85] and interval[ARBE92]. These ambiguity expressions have their merits and demerits, which are summarized in Table 1. As a new mode of ambiguity expression for the pairwise comparison measurement we will propose to use "complex number", instead of "real number". Although its measurement may not be so easy compared to the probabilistic comparison or the interval comparison, the C-comparison has its advantages in transitivity satisfaction in consistency case and understanding its process and result, whose explanation will be tried.

Measuring pairwise comparison judgement in complex number

With the conventional fundamental scale, the intensity of importance ranges from 1 to 9. Say the importance intensity "5" (strong importance) means that experience and judgment strongly favor one activity over another. Mathematically speaking it means that experience and judgment 5 times more favor one activity over another. Here all the pairwise comparison judgments are measured in the real. If we have some kind of ambiguity with this intensity "5" measurement, how do we distinguish among them. Then, complex number a_{jk} will be proposed to distinguish among the measurements with the same intensity but different ambiguity degrees.

$$a_{jk} = r_{jk} \exp(i\theta_{jk}) \qquad (1)$$
$$= r_{jk} (\cos\theta_{jk} + i \sin\theta_{jk}) \qquad (2)$$

r_{jk} : intensity or amplitude of (j, k) pairwise comparison judgment

θ_{jk} :degree of inclination angle from the real toward the imaginary
i = imaginary unit

Table 1 Merits and demerits of ambiguity expressions

	Probabilistic comparison	Fuzzy comparison	Interval comparison
Measurement	Easy	Moderate	Easy
Weight estimation	Moderate	Difficult	Difficult
Process acountabilty	Moderate	Questionable	Questionable
Validity of result	Moderate	Questionable	Questionable
Transitivity satisfaction in consistency case	Moderate	Difficult	Difficult

3. Estimating complex priority weight (C-weight) from C-comparison matrix

Two C-weight estimation methods are introduced in this section. They are the power method and the geometric mean method. With the power method, C-weight w is calculated by Eq.(3).

$$w = \lim_{p \to \infty} \frac{A^p e}{e^T A^p e} \qquad (3)$$

Here, A is a C-comparison matrix and e is an all-1 column vector of appropriate size. With the geometric mean method, C-weight w is calculated by Eq.(4).

$$w_j = \frac{(\prod_{k=1}^{n} r_{jk} e^{i\theta_{jk}})^{\frac{1}{n}}}{\sum_{j=1}^{n} (\prod_{k=1}^{n} r_{jk} e^{i\theta_{jk}})^{\frac{1}{n}}} \qquad (4)$$

Next we will show two numerical examples.

Example 1

Consider a C-comparison matrix of size 3 given by Eq.(5).

$$A=\begin{bmatrix} 1 & 3e^{i\frac{p}{18}} & 4e^{i\frac{p}{9}} \\ \frac{1}{3}e^{-i\frac{p}{18}} & 1 & 5e^{i\frac{p}{12}} \\ \frac{1}{4}e^{-i\frac{p}{9}} & \frac{1}{5}e^{-i\frac{p}{12}} & 1 \end{bmatrix} \qquad (5)$$

Then, its C-weight vector estimated by the power method is given by Eq.(6).

$$w=\begin{bmatrix} 0.5997e^{i\cdot 0.081} \\ 0.3106e^{i\cdot(-0.065)} \\ 0.0965e^{i\cdot(-0.297)} \end{bmatrix} \qquad (6)$$

Its C-weight vector estimated by the geometric mean method is given by Eq.(7).

$$w=\begin{bmatrix} 0.5997e^{i\cdot 0.081} \\ 0.3106e^{i\cdot(-0.065)} \\ 0.0965e^{i\cdot(-0.297)} \end{bmatrix} \qquad (7)$$

As can be seen from Eqs.(6) and (7), the two estimated C-weight vectors coincide exactly.

Example 2

Consider a C-comparison matrix of size 4 given by Eq.(8).

$$A=\begin{bmatrix} 1 & 4e^{i\frac{p}{3}} & 6e^{i\frac{p}{4}} & 5e^{i\frac{p}{2}} \\ \frac{1}{4}e^{-i\frac{p}{3}} & 1 & 3e^{i\frac{p}{7}} & 2e^{i\frac{p}{5}} \\ \frac{1}{6}e^{-i\frac{p}{4}} & \frac{1}{3}e^{-i\frac{p}{7}} & 1 & 7e^{i\frac{3}{7}p} \\ \frac{1}{5}e^{-i\frac{p}{2}} & \frac{1}{2}e^{-i\frac{p}{5}} & \frac{1}{7}e^{-i\frac{3}{7p}} & 1 \end{bmatrix} \qquad (8)$$

Its C-weight vector estimated by the power method is given by Eq.(9).

$$w = \begin{bmatrix} 0.6406e^{i\cdot 0.328} \\ 0.2188e^{i\cdot(-0.355)} \\ 0.1831e^{i\cdot(-0.335)} \\ 0.0719e^{i\cdot(-1.354)} \end{bmatrix} \qquad (9)$$

Its C-weight vector estimated by the geometric mean method is given by Eq.(10).

$$w = \begin{bmatrix} 0.6823e^{i\cdot 0.368} \\ 0.2281e^{i\cdot(-0.475)} \\ 0.1628e^{i\cdot(-0.454)} \\ 0.0713e^{i\cdot(-1.369)} \end{bmatrix} \qquad (10)$$

As can been seen from Eqs.(9) and (10), the two estimated C-weight vectors are slightly different. They do not coincide in this case of size 4.
Based on these numerical examples, we can say that, for a general, not necessarily consistent, C-pairwise matrix, the two estimated C-weight vectors are the same in the case of size 3 and they can be different in the case of size 4 or more.

4. Complex AHP vs Real AHP

Through a car-selection AHP example of Fig.1, we will illustrate how C-comparison is incorporated in Complex AHP.

Their real-valued comparison matrices are given in Table 2, where the judgment is considered crisp with no ambiguity. The priority weight vector for the criteria is given by Eq.(11).

$$w_c = \begin{bmatrix} 0.5437 \\ 0.3109 \\ 0.0974 \\ 0.0479 \end{bmatrix} \qquad (11)$$

The priority weight vector for the alternatives, A, B and C, is given by Eq.(12).

$$w_A = \begin{bmatrix} 0.388417 \\ 0.428022 \\ 0.183561 \end{bmatrix} \qquad (12)$$

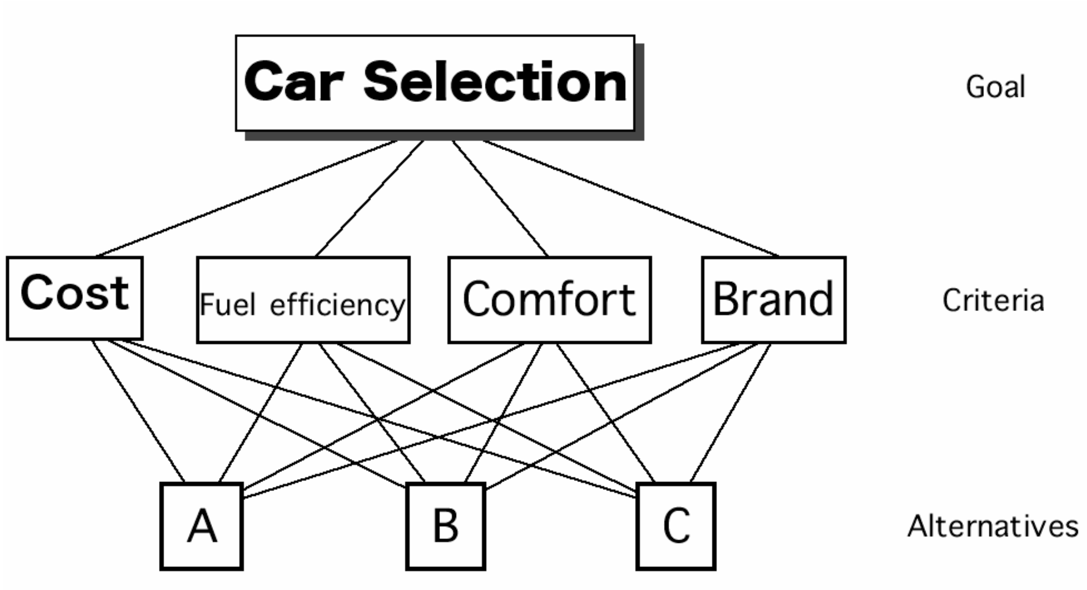

Fig.1 AHP diagram of car selection problem

Next, the value of a_{13} is changed from 5 to 5 $\exp(i\frac{p}{4})$ in the pairwise comparison matrix A among the criteria (Table 3). This means that some ambiguity of inclined angle= 45° is considered in the judgment of a_{13}. Then, the priority weight vector for the criteria is given by Eq.(13).

$$w_c^{*} = \begin{bmatrix} 0.5442e^{0.1i} \\ 0.3172e^{-0.095i} \\ 0.0967e^{-0.246i} \\ 0.0489e^{-0.017i} \end{bmatrix} \qquad (13)$$

The priority weight vector for the alternatives is given by Eq.(14).

$$w_A^{*} = \begin{bmatrix} 0.386198e^{0.034523i} \\ 0.430908e^{-0.024113i} \\ 0.183273e^{-0.016046i} \end{bmatrix} \qquad (14)$$

5. Conclusion

In order to express some kind of ambiguity in the decision making, we propose to use complex number pairwise comparison (C-comparison) and complex number AHP (C-AHP). C-comparison and C-AHP are illustrated through some examples. Validity and applicability of C-comparison and C-AHP still remain complex and uncertain, which are future research subjects.

Table2 : Pairwise Comparison Matrices

Criterion	Cost	Fuel	Comfort	Brand
Cost	1	3	5	7
Fuel	$\frac{1}{3}$	1	5	7
Comfort	$\frac{1}{5}$	$\frac{1}{5}$	1	3
Brand	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{3}$	1

Cost	A	B	C
A	1	2	3
B	$\frac{1}{2}$	1	2
C	$\frac{1}{3}$	$\frac{1}{2}$	1

Fuel	A	B	C
A	1	$\frac{1}{5}$	$\frac{1}{2}$
B	5	1	7
C	2	$\frac{1}{7}$	1

Comfort	A	B	C
A	1	3	2
B	$\frac{1}{3}$	1	$\frac{1}{2}$
C	$\frac{1}{2}$	2	1

Brand	A	B	C
A	1	$\frac{1}{2}$	$\frac{1}{2}$
B	2	1	1
C	2	1	1

Table3 : C-comparison Matrix

Criterion	Cost	Fuel	Comfort	Brand
Cost	1	3	$5e^{i\frac{\pi}{4}}$	7
Fuel	$\frac{1}{3}$	1	5	7
Comfort	$\frac{1}{5}e^{-i\frac{\pi}{4}}$	$\frac{1}{5}$	1	3
Brand	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{3}$	1

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