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ORDER STABILITY ANALYSIS OF RECIPROCAL JUDGMENT MATRIX

Mingzhe Wang[a] Danyi Wang[b]
[a] Huazhong University of Science and Technology, Wuhan, Hubei 430074 - P. R. China
mzwangpublic.wh.hb.cn
[b] George Mason University, Fairfax, VA 22030 USA, dwang2@gmu.edu

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Summary: *Various kinds of pairwise comparison judgment matrixes are the basic components in the Analytic Hierarchy Process (AHP) Frame for multicriteria decision making. This paper provides the concepts of rank equivalent classes and order stable structure of the positive reciprocal judgment matrix in AHP. It is shown that the appropriate order stability structure of judgment matrix is a basic and important condition in decision judgment and analysis process to preserve the reliability of priority order ranking derived from the judgment matrix. And also, in this paper, a simple method to detect the unstable order structure of matrix is presented.*

1. Introduction

The analytic hierarchy process (AHP) introduced by T.L. Saaty is a well-known and popular analytic model of multi-criteria decision making. The final object of AHP is the priority ordering of the decision alternatives, however, the first and basic concern in AHP is to obtain a rank ordering of the priority vector from the positive reciprocal judgment matrix under a single criterion. We look on each column vectors expressed in the judgment matrix as one time of judgment by decision maker and also one element in positive vector space. Due to a lot of uncertain factors and perturbations, the n column vectors in the matrix are absolutely not consistent. So, people have to look for the column vectors in the matrix is a " close" structure to compose a close consistent matrix. The " close" here means the "order clustering" that all column vectors of a judgment matrix have a same (or near same) rank ordering sequence.

Let us suppose that an n by n pairwise comparison judgment matrix A=(a_{ij}) in AHP is positive reciprocal, i.e. matrix A holds a_{ij} =1/ a_{ji} , a_{ij} >0. It is consistent if $a_{ik}a_{kj} = a_{ij}$, i, j, k? (1, 2, … , n-1). A lot of works respect to the issue of the near consistent level of a judgment matrix have been done. Saaty and Vargas (1982, 1988, 1995) discussed the test standard of consistency (consistency ratio, CR=(l_{max}-n) / (n-1); Random Consistency Index RI) and suggested the near consistency level to be : CI/RI= 10% (5% and 8% only for three and four alternatives). However, it is clear that a decision making process heavily depends on the rank ordering of the judgment for alternatives, associated with inconsistency, the inverse ranking inconsistency is a troublesome problem. One aspect of the problem is how serious the priority ordering of judgments was distorted. In recent years, Aguaro n (2000) [1], Hurley (2001) [2], Lipovetsky (2002) [3] and Saaty (2003) [4] have studied and presented some methods to detect the distorted entries and to look for new one instead in order to improve the consistency of judgment matrix. Although all of those works are effective, from the view of rank order structure of

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judgment matrixes, these approaches do not catch the fancy of the favorable rank ordering for the judgment matrix completely.

For examples, let A and B be 3×3 and 4×4 positive reciprocal pairwise comparison judgment matrixes. The estimated results of priority vectors with rank order numbers of A and B are in table1 and 2 by using the eigenvector method (EM), the normalization of row average (NRA), the least squares method (LSM) and the logarithmic least square method (LLSM) on the matrix A and B respectively.

Table 1: Different priority vectors and rank order of matrix A

A	A_1	A_2	A_3	EM	NRA	LSM	LLSM
A_1	1	1/4	1/3	0.1261 **(1)**	0.1253 **(1)**	0.1258 **(1)**	0.1291 **(1)**
A_2	4	1	4/5	0.4254 **(2)**	0.4591 **(3)**	0.4025 **(2)**	0.4355 **(2.5)**
A_3	3	5/4	1	0.4485 **(3)**	0.4156 **(2)**	0.4717 **(3)**	0.4355 **(2.5)**
	l_{max} =3.0291, C R = 0.028 < 5%						

Table 2: Priority vectors and rank order estimate by using various methods for matrix B

B	B_1	B_2	B_3	B_4	EM	NRA	LSM	LLSM
B_1	1	1/2	1/4	3/2	0.1499 **(2)**	0.1592 **(2)**	0.1346 **(1)**	0.1493 **(2)**
B_2	2	1	1	3	0.3432 **(3)**	0.3429 **(3)**	0.3642 **(4)**	0.3550 **(3.5)**
B_3	4	1	1	3/2	0.3635 **(4)**	0.3673 **(4)**	0.3538 **(3)**	0.3550 **(3.5)**
B_4	2/3	1/3	2/3	1	0.1434 **(1)**	0.1306 **(1)**	0.1475 **(2)**	0.1407 **(1)**
	l_{max} =4.1885, C R = 0.069 < 8%							

Note: in table 1 and 2, the order number of every element in the priority vectors is given in parentheses, and the elements with same rank in a priority vector are marked with an average order value of their rank numbers. Especially note: the order numbers in the tables are just presented as marks of rank orderings, which do not reflect intensity priority ratios between two adjacent elements in vector or two adjacent ranked alternatives.

According to the consistency ratio C.R., obviously, the consistency levels of matrixes A and B are accepted. But their rank orderings of the different priority vectors estimated by methods are not same, especially some ranks of alternatives are reversed. Which priority vector among them is decision maker look for? As Professor Saaty suggested, the principal eigenvector calculated by EM is necessary for representing the priorities derived from the near consistent judgment matrices

2. Order Structure of Judgment Matrix

An important basic aspect of AHP is the estimation of the priority rank ordering of the alternatives from the judgment matrix. The general form of a judgment matrix is a positive reciprocal square matrix, which contains n un-normalized column vectors. In order to represent the decision maker's ordinal perception faithfully, let us consider the strict preference order relations such as $A_1 \succ A_2 \succ \dots A_i \succ \dots \succ A_n$ and the same (or indifferent) preference order such as $A_k \sim A_{k+1}$, k? (1, 2, … , n-1) for the set of alternatives {A_1 , A_2, … A_i, …, A_n }. One column vector of a judgment matrix A=(a_{ij}) is regarded as one time of judgment with the different unitage alternative A_i, i? (1, 2, … , n) .

We assume that any column vector a_j of a judgment matrix is an element of the vector space R_n^+, where R_n^+ ={ a_j = $(a_{1j}, a_{2j}, \dots , a_{nj})^T$ | a_{ij} >0, i, j ? (1, 2, …, n) }. Then, a strict preference order $A_i \succ A_j$ holds a_{ij} >1 and same preference order $A_i \sim A_j$ holds a_{ij} =1 in the matrix $A = (a_{ij})$ .Sometimes, we concisely use the natural number sequence to symbolize the strict rank orderings, and use the average order value of their rank numbers to characterize the same rank positions, such as presented in table1 and 2.

It is easy to know that each column vector of the judgment matrix can described as once judgment and a vector a_j ? R_n^+. Obviously, with any a ? R_1^+, two vectors of a_j and a a_j present a same priority ordering, in other words, a a_j does hold not only the same rank ordering but also the same strength ratio for compared alternatives in despite of any different a ? R_1^+. Specially, it is easy to understand that all of the column vectors in a judgment matrix can be expressed as one vector with different coefficient a >0 if and only if the matrix is consistent. It is also true that all of column vectors in a consistent matrix indicate same priority rank ordering and intensity ratio for decision alternatives.

Definition 1. Any column vectors of matrix $A = (a_{ij})$, such as column vectors $a_i, a_j \in R_n^+$, have same priority rank order if for each k, l ? (1, 2, … , n), $a_{ki} > a_{li}$ and $a_{kj} > a_{lj}$ or $a_{ki} \approx a_{li}$ and $a_{kj} \approx a_{lj}$, that means they have equivalence order relation EO, denoted by a_i EO a_j.
For example: a_2 EO a_3 in table 1.

Theorem 1. EO is a binary relation of R_n^+. It is also a rank equivalence relation that determines unique rank equivalent class of R_n^+.

Proof. See [5].

Definition 2. A rank equivalent class with k rank positions in same order position is called a (n-k-1) -dim rank equivalent class. The dimension of a strict rank equivalent class in R_n^+ is n.

Theorem 2. The vector space R_n^+ holds n! strict rank equivalent classes. Each one of strict rank equivalent classes adjoins (n-1) side rank equivalent classes in which only one couple rank positions holds same rank order, (n-2) side rank equivalent classes with only three same rank positions, (n-k) side rank equivalent classes with (k+1) same rank positions, and so on.

Proof. Firstly, let us show the amount of the strict rank equivalent classes in R_n^+. Given a strict rank equivalent class as a set of vectors of R_n^+, without loss of generality, it represents $A_1 \succ A_2 \succ \dots A_k \succ \dots \succ A_n$ and denoted by the natural number sequence $(1, 2, \dots , n)^T$. Hence, its amount is equal to the number of maps of itself, so that "1" position has n images of the maps, "2" position has (n-1) images, "3" has (n-2) images, and so on. Therefore, the amount of these strict rank order classes in R_n^+ is n!. Secondly, let us fix the rank ordering of a certain strict rank equivalent class and consider the same positions as one rank position, it is easy to show that the amount of this side rank equivalent classes with two same rank positions is (n-1), the side equivalent classes with three same rank is (n-2), and so on. Especially, there exists only one side rank equivalent class in which whole rank positions are with same order number in R_n^+. And obviously, any vector of R_n^+ must belong in one rank equivalent class.

Definition 3. Based on one rank equivalent class, its neighbor rank equivalent class is obtained by exchanging only one couple of connecting order positions. Its side rank equivalent class is obtained by extending the same order positions without changing its other rank orderings, and all of its side rank equivalent classes construct its bound rank equivalent class.

Theorem 3. A certain strict rank equivalent class represented alternative ordering as $A_1 \succ A_2 \succ \ldots A_k \succ \ldots \succ A_n$ is a n-dim rank equivalent class, which has 2^{n-1} of side equivalent classes to be composed its bound equivalent class in R_n^+.

Proof. By similar way of the proof for theorem 2, the result of theorem 3 is obvious.

For instance, based on strict rank equivalent class associated with $A_1 \succ A_2 \succ \ldots A_k \succ \ldots \succ A_n$. It is a n-dim rank equivalent class denoted by (1, 2, … , n). Its bound rank equivalent class includes all of its side rank equivalent class, that is, the (n-1) of (n-1)-dim side rank equivalent classes with two same rank positions, (n-2) of (n-2)-dim side equivalent classes with three same rank positions, and so on. The total number of side equivalent class in a bound class is equal to 2^{n-1}. A strict rank equivalent class has (n-1) neighbor rank equivalent classes.

It is easy to imagine that any rank equivalent class in R_n^+ is a cone. Moreover, we will prove later that it is also a convex cone. A certain rank equivalent class and its bound equivalent class together compose a convex cone too in R_n^+, called quasi-close rank equivalent class. This is an important region specially which is constructed with a strict rank equivalent class.

Definition 4. A quasi-close rank equivalent class is a set of this rank equivalent class with its bound equivalent class together.

An illustrative example, let n=2, the R_2^+ space is a plane divided into three rank equivalent classes by the 45^0 line. All vectors above the 45^0 line in the plane belong into one rank equivalent class represented $A_1 \succ A_2$, all vectors below the 45^0 line belong into another rank equivalent class represented $A_2 \succ A_1$, and the 45^0 line represented $A_1 \sim A_2$ is only one side rank equivalent class which composes a bound class both of previous equivalent classes.

Theorem 4. Any rank equivalent class in the vector space R_n^+ is a convex cone.

Proof. Actually, according to the definition of convex cone that a cone is a convex cone if and only if $l_1 x_1 + l_2 x_2 \in K$ whenever $x_1 \in K$, $x_2 \in K$, l_1>0 and l_2>0. The first is obvious that each rank equivalent class K of R_n^+ is a cone. And second, by the definition2 in the paper, since for any vector x_1 and x_2 ? K, if $x_{k1} \geq x_{l1}$ and $x_{k2} \geq x_{l2}$, k, l ? (1, 2, …, n), the sum vector $l_1 x_1 + l_2 x_2$ has $l_1 x_{k1} + l_2 x_{k2} \geq l_1 x_{l1} + l_2 x_{l2}$ for any l_1>0, l_2>0, so the sum vector must belong to the same rank equivalent class K. It means that any rank equivalent class $K \subseteq R_n^+$ is a convex cone.

Naturally, the results of theorem 4 can be extended such to any linear combination of vectors in a rank equivalent class. Based on this conclusion and the knowledge of linear space structure, we have a corollary for judgment matrix as follows.

Corollary 1. All independent column vectors of a judgment matrix compose a convex cone in R_n^+.

3. Order Stability Region of Judgment Matrix

In fact, Analytic Hierarchy Process (AHP) provides a well-known cognitive control mode to help people make complex decisions. It also allows for inconsistency of judgment matrix since there are many uncertain factors and perturbations in making judgments. No matter how precise it is, as Saaty said [4], if one insists on consistency, people would be required to be like robots, but if no rule to aid complex judgments, people would arbitrarily change their minds to be like drunkards. In order to find some useful guidelines embedded in the frame of analytic hierarchy process to support people endeavor, in this section, we will analyze the stable structure of judgment matrixes and find the stable region for decision alternatives in this section.

Definition 5. The order structure of a judgment matrix is said to be stable if any no-negative linear combinations of all column vectors of this judgment matrix keeps same rank order for decision alternatives.

One column vector of the judgment matrix exhibits once judgment. If the matrix order structure is stable, then how big weight put on some of column vectors of the matrix is unable to change the rank ordering of the final priority order for alternatives under a single criterion

It is clear that the order structure of a consistent judgment matrix is stable since all column vectors of this kind of judgment matrix belong to one ray in a positive vector phase.

Theorem 5. A sufficient condition for stable order structure of judgment matrix is that all of its column vectors belong to one rank equivalent class in the vector space R_n^+

Proof. As mentioned above, since a rank equivalent class is a convex cone, any one of the no-negative linear combinations of those column vectors must be in this rank equivalent class if all column vectors of judgment matrix are in this rank equivalent class. Therefore, the combination of column vectors, as a priority vector estimated from the judgment matrix, must also belongs to the same rank equivalent class. Actually the combination vector keeps same rank ordering as the column vectors hold. This kind of matrix is order structure stable.

More detail, supposing combinations $l_1 a_1 + l_2 a_2 + \cdots l_j a_j + \cdots + l_n a_n$ where a_j is a column vector of the judgment matrix, $l_j = 0$, j? (1, 2, … , n). Supposing all of column vector a_j are in one rank equivalent class, that is all vectors have same priority rank ordering for alternatives, i.e., for any index k and l and any column vector of matrix, let a_j and a_{j+1}, if $a_{kj} \geq a_{lj}$ then $a_{kj+1} \geq a_{lj+1}$, k, l ? (1, 2, …, n), so we have the combination $\sum_{j=1}^{n} l_j a_{kj} \geq \sum_{j=1}^{n} l_j a_{lj}$ k, l ? (1, 2, …, n) belongs to a same rank equivalent class.

According to the definition 5, we have more corollary as follows.

Corollary 2. (1) The order structure of a positive reciprocal judgment matrix is strict stable if all of its column vectors belong to one strict rank equivalent class of R_n^+

(2) The order structure of a positive reciprocal judgment matrix is critical stable if all of its column vectors are in a side rank equivalent class of R_n^+

(3) The order structure of a positive reciprocal judgment matrix is stable if all of its column vectors are in a quasi-close rank equivalent class of R_n^+

(4) The order structure of a positive reciprocal judgment matrix is asymptotically stable if its dimension of the matrix is adequate and all of its column vectors are distributed in the region of a rank equivalent class with its adjoining classes.

4. Stability Analysis Based On Examples

With regard to stability analysis of the judgment matrix, the first task is to identify the components of the rank equivalent classes of the matrix and determine the validity of its order stable structure. If the order structure is not accepted, we need to deal with the relationship between changes in the judgment vectors, and arrange the rank reversal of the alternatives.

Now, let consider the example of judgment matrix A, its three column vectors represent $A_3 \succ A_2 \succ A_1$ and $A_2 \succ A_3 \succ A_1$ in the two strict rank equivalent classes. Both classes adjoin each other. So, the order structure of matrix A is unstable due to n=3 and it will be better to ask decision maker to judge it.

By using similar way, the four column vectors of matrix B are in the three different rank equivalent classes, ? $B_3 \succ B_2 \succ B_1 \succ B_4$, ? $B_3 \sim B_2 \succ B_1 \succ B_4$ and ? $B_2 \succ B_3 \sim B_1 \succ B_4$ respectively. The second class is the first one's side rank equivalent class, and also the second class is the side class between the first class and first one's neighbor rank equivalent class ($B_2 \succ B_3 \succ B_1 \succ B_4$). Specially, the third class is the side class of the first one's neighbor rank equivalent class, but it is broke away by a strict class. So, it is so far from the region of the most of the rank classes that the judgment matrix holds. Consequently, the order structure of matrix B is also unstable and it is necessary to ask the decision maker to change his mind, first to third one.

In practice, the rank order numbers of vectors provide a useful way to help people to determine whether or not the order structure stability of judgment matrix is satisfied. For example, the rank order numbers of the column vectors in matrix A and B are presented as follows:

$$A: \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix} \quad B: \begin{pmatrix} 2 & 2 & 2 & 2.5 \\ 3 & 3.5 & 3.5 & 4 \\ 4 & 3.5 & 3.5 & 2.5 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

5. Conclusion And Future Research

In this paper, we have discussed the concepts of rank order equivalent class, configurations of rank equivalent classes in vector space and stability structures of judgment matrix. Using these concepts, we can analyses the order structure of judgment matrixes, change the rank reversal in the judgments and improve the validity of the priority vector for decision alternatives. The satisfactory order structure of a judgment matrix is an essential condition to preserve the rank ordering of solution in AHP. The analyzing process and methods as we mentioned above are convenient, simple and useful to estimate a priority rank ordering to the best of the decision maker's mind for multi-criteria decision making problems.

However, one substantial effort that we did not consider in this paper is that the rationality of rank order structure of people's judgments and expressions in positive vector space. And also, human error and various uncertain perturbations may exist in each facet of analysis process. Analytic Hierarchy Process provides a very helpful cognitive control mode to implement and preserve quality of solutions of multi-criteria decision making, but complexity of real cognitive situation is a truly very large problem. We have not solved it. Hopefully, we have set forth a good start for solution in this field.

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