MUDDLED MAGNITUDES

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Summary: *A problem with a known composite answer is used in conjunction with different normalization routines to determine correct procedures for generating criteria weights. The method of normalization and the criteria weights are closely associated. Alteration of one without corresponding adjustment of the other can lead to incorrect overall ratios.*

1. Introduction

In the additive AHP model with relative measurement, the overall preferences $V_1,V_2,...,V_n$ of $A_1,A_2,...,A_n$ are estimated by the weighted arithmetic means $f_1,f_2,...,f_n$

$$f_j = w_1ß_1y_{j1} + w_2ß_2y_{j2} + ... + w_mß_my_{jm}\ ,\ j = 1,2,...,\ \ n \qquad (1)$$

where

w_i	is the importance weight of criterion i
y_{ji}	is a ratio derived scale that measures and estimates alternative A_j on criterion i
$ß_i$	is a positive constant that represents different scaling or normalizations of the ratio y_{ji}

The f_j estimates are in ratio form to the overall preference. In other words, $f_1/.f_2 = V_1/V_2$. The values of y_{ji} are usually not known explicitly and a pairwise comparison matrix of preferences is used to estimate the values of y_{1i}, y_{2i},...,y_{ni} for each criterion i. We note that the unknown values y_{1i}, y_{2i},...,y_{ni} are indefinite insofar as they are ratio numbers that can be multiplied by any positive constant. Initially, that fact is not important, because only ratios among y_{1i}, y_{2i},...,y_{ni} are important in AHP computations.

The scaling constants $ß_1$, $ß_2$,..., $ß_m$ in (1) have been included explicitly to show that a positive multiple does not destroy the ratio relationship amongst a criterion's y_{ji}. Usually, $ß_i$ is chosen so that the $ß_iy_{ji}$ of all criteria conform to the same standard format. For example, the AHP distributive mode normalizes y_{ji} for each criterion so that the resulting local priorities ($ß_iy_{ji}$) across alternatives sum to one. However, in the ideal mode, the standard format utilizes a normalization that makes the local preference of the best alternative equal to 1 and the sum of $ß_iy_{ji}$ >1. Both sets of $ß_iy_{ji}$ are in a standard ratio form for their respective modes, but they have different values because they have different ß.

From the same perspective, the w_i in (1) can be viewed as scaling constants or weights that transform the $ß_iy_{ji}$ values into commensurate units that can be aggregated through addition. In order for the f_j values to

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be in ratio form consistent with V_1, V_2, … V_j, the partial $w_i ß_i y_{ji}$ values must be commensurate across criteria. The criteria weights assure that property.

Since any positive ß maintains the ratio relationship amongst the y_{ji} of a criterion, it is obvious that different values for $ß_1, ß_2, ..., ß_m$ will require different values of w_i to maintain commensurate properties. It also means that $ß_1, ß_2, ..., ß_m$ must be specified in advance or known in (1) before determining the criteria weights $w_1, w_2, ..., w_m$.

The prime purpose of this paper is to investigate how w_i must change when different ß are used for each criterion. In effect, we intend to muddle the magnitudes of each criterion's priorities and then see how criteria weights must be adjusted to maintain commensurate relationships. It should be noted that such muddling ought not be undertaken in actual practice. Nevertheless, such muddling helps us to understand correct procedures in practice.

2. Procedure

Our procedure is to take a problem with a known true values for $V_1, V_2, ..., V_n$ and then use different ß to determine what w_i would have to be to maintain those answers. In essence, we are reverse engineering the AHP process in order to get the correct answer. The purpose is to find out how the process works to get correct estimation rather than carrying out the estimation itself.

Calculations for problems with known answers have been used before to authenticate AHP procedures. Saaty (1977) and Saaty and Vargas (1991) validated the eigenvector routine by measuring distances from Philadelphia, intensities of lights, and areas of objects. For combining eigenvectors into a composite solution, several researchers have used problems with known answers. Schoner and Wedley (1989) used a car purchase example. Wedley et al (1993, 1996) used multiple distances from Singapore. Vargas (1997) used a known answer problem to show that multiplicative composition gives rise to invalid answers. Similarly, Saaty (1999, 2001) uses known purchase prices and remodeling costs of houses to show that multiplicative synthesis does not yield the correct composite ratio.

Rather than select a new situation with known answers, this study uses Vargas' (1997) example of 3 boxes that have different components of 4 objects. Overall preference is measured by the total weight of each object after the 3 components have been removed from the boxes and assembled. Since the weight of each component in each box is known, the true weights of each object can be ascertained. The object weights in each box and the overall true priorities are given in Table 1. We note that the weights of all objects in all boxes are measured in commensurate units with object 1 in box1 representing the unity of one pound. Thus the row totals correctly represent the weights of the objects. Like Vargas, we choose the ability of a method to replicate the true composite ratios as the measure of effectiveness.

Table 1 – Weights (lbs.) of Object Components in Three Boxes

	Box 1	Box 2	Box 3	**Total**	**True relative priorities**
Object 1	1	6	10	**17**	0.243
Object 2	2	4	14	**20**	0.286
Object 3	3	8	6	**17**	0.243
Object 4	4	2	10	**16**	0.229
Total	**10**	**20**	**40**	**70**	1

It is convenient to view the problem in the following manner. Assume that a person is given three boxes and is told that there are components of 4 objects inside each box and that each component will be assembled to make the object. In decision analysis terminology, the assembled objects to be measured are the alternatives and the boxes are the component criteria. The person is told that the goal is to estimate what the relative weights of the objects will be if they are assembled. The person is advised that once the boxes are opened, comparisons can be made between components to determine relative priorities within boxes (i.e. y_{ji} values). However, the person is also told that different $ß_i$ values will be used to rescale those within-box priorities to a different unit. The person's problem is then to generate criteria weights (w_i) that will generate overall preferences that are in ratio form to the true relative priorities. For illustrative purposes, we assume that the person does not have a mechanical scale, but can make comparisons judgments between any two items with perfect accuracy.

2.1 Estimating y_{ji} values

Once the boxes are opened, there are many possible ways that the person can estimate relative priorities for the items within each box. For example, the person may take the component of Object 1 as the reference or unit of measure for each box. The resulting relative priorities would be as shown in Table 2. Alternatively, the person could take the component of Object 1 in the Box 1 as the reference or unit of measure for all boxes. The resulting relative priorities would be the weights shown in Table 1. If the person knew how to extract the local priorities from a matrix of paired comparisons, then that process would also yield sets of relative priorities in ratio form. Whatever the process, it should be noted that many different ratio sets are possible

Table 2 – y_{ji} values --Box $_i$ items Relative to Object 1's item

	Box 1	Box 2	Box 3	Simple Total
Object 1	1	1	1	**3**
Object 2	2	0.667	1.4	**4.0667**
Object 3	3	1.333	0.6	**4.9333**
Object 4	4	0.333	1	**5.3333**
Total	**10**	**3.333**	**4**	**17.333**

In effect, the values of y_{ji} are indefinite. The only requirement is that they have a ratio form that can be rescaled to another form while maintaining ratio relationships within each column. Because there are many different possible ratio sets, we will assume that our decision maker generates the set shown in Table 2. We note that any other form that maintains the correct ratios would be acceptable for Table 2. For example, the usual eigenvector result that normalizes columns to sum to one would have been just as appropriate. The values chosen for display in Table 2 emphasize that only the ratio relationships are important for y_{ji} values of each column.

Notice in Table 2 that the column of simple totals of local y_{ji} priorities does not yield the correct aggregation ($4.0667/3 \neq .286/.243 = 1.176$). That is because the columns are not in commensurate units. In order to do that, we need to use appropriate w_i values as scaling constants. But before that is done, AHP procedures such as the distributive and ideal modes use $ß_i$ values to rescale y_{ji} values to respective standard formats.

2.2 The Distributive Mode

Before aggregation, the distributive mode utilizes $ß_i$ values that normalize the alternative priorities of each criterion to sum to one. Hence, $ß_i = 1/\Sigma y_{ji}$, and the columns of y_{ji} values in Table 2 are normalized by the sum of the columns to yield Table 3.

Table 3 – Distributive mode $ß_iy_{ji}$ and Composite Priorities

Criteria priorities=	1	2	4		
	Box 1	Box 2	Box 3	**Simple Total**	**Weighted Total**
Object 1	0.1	0.3	0.25	**0.65**	1.700
Object 2	0.2	0.2	0.35	**0.75**	2.000
Object 3	0.3	0.4	0.15	**0.85**	1.700
Object 4	0.4	0.1	0.25	**0.75**	1.600
Total	**1**	**1**	**1**	**3**	

Notice that the simple totals of $ß_iy_{ji}$ values, like y_{ji} values, do not yield the correct aggregate priorities: (.75/.65 $\neq$ .286/.243 = 1.176). That is because the $ß_i$ values merely rescale the columns to sum to one rather than being commensurate across the columns. It is still the function of the w_i values to transform the columns to commensurate units before addition. That transformation can be achieved by any set of criteria weights that captures the overall weight between boxes. In Table 3, we have used the total or average weight of Box 1 as the unit of comparison for criteria weights. We could have just as easily used any other box as the reference unit or we could have normalized those weights to sum to one (as is the usual method in AHP). Whichever method using totals or averages, the aggregate priorities yield the correct relative overall preferences (2/1.7 = 1.176).

2.3 The Ideal Mode

The ideal mode of relative measurement does not use the same $ß_i$ values as the distributive mode for its standard format. Instead of normalizing so that the sum of $ß_iy_{ji}$ across alternatives equals 1, the ideal mode normalizes the best alternative under each criterion to unity. Hence, $ß_i$ = 1/best y_{ji} which means that the best alternative under each criterion equals unity. All other alternatives assume a $ß_iy_{ji}$ value less than one. Scaling the y_{ji} of Table 2 in this manner yields Table 4.

Table 4 – Ideal mode $ß_iy_{ji}$ and Composite Priorities

Criteria priorities=	0.5	1	1.75		
	Box 1	Box 2	Box 3	**Simple Total**	**Weighted Total**
Object 1	0.25	0.75	0.714	**1.7143**	2.125
Object 2	0.5	0.5	1	**2**	2.500
Object 3	0.75	1	0.429	**2.1786**	2.125
Object 4	1	0.25	0.714	**1.9643**	2.000
Total	**2.5**	**2.5**	**2.857**	**7.8571**	

It should be noted that the unit of measure for the columnar $ß_iy_{ji}$ values is the ideal alternative of each column and that those units of measure are incommensurate with one another (e.g. 1 unit of object 4 in Box 1 $\neq$ 1 unit of Object 3 in Box 2). Commensurability is achieved if the w_i convert the unit of each column to a new unit that is common across criteria. This fact indicates that a correct set of w_i will be found if our analyst makes comparisons to get relative ratios between the ideal alternatives that are the units for each column. We have done that in Table 4, using Object 3 of Box 2 (an 8 lb. item) as our unit of comparison across ideal components of different boxes. Thus, the ideal of Box 1 (4 lbs.) is .5 of the referent ideal and the ideal of Box 3 (14 lbs.) is 1.75 the ideal of Box 2

In generating criteria weights, we note that it is only important to get the relative weights across of the ideal referents of each column that have been assigned the unity $ß_iy_{ji}$ values. Any of the other ideal alternatives besides Box 2 could have been selected as the referent for comparisons. And as noted for the distributive mode, the resulting criteria weights could have been normalized to sum to one. We have chosen to leave criteria weights normalized to the Box 2 ideal as unity. The resulting weighted priorities take the correct relative overall preferences (2.5/2.125 = 1.176).

2.4 Muddled mode.

So far, we have illustrated the standard formats of the distributive and ideal modes whereby the same $ß_i$ formula ($1/\Sigma y_{ji}$ or $1/\text{best } y_{ji}$) is used for each column. But since the ratio relationships within a column are maintained with any positive $ß_i$, it is possible to use a mix of normalization formulas across columns. Although permissible, this muddling of magnitudes makes it more difficult for our analyst to determine appropriate w_i values.

Table 5 presents muddled magnitudes in each column. The y_{ji} for each column in Table 2 have been normalized in the following manner: for Box 1,according to the distributive mode (i.e. $ß_1 = 1/\Sigma y_{j1}$); for Box 2, according to the ideal mode (i.e. $ß_2 = 1/\text{best } y_{j2}$); and for Box 3, according to the least desirable alternative (i.e. $ß_3 = 1/\text{worst } y_{j3}$). From the preceding, it should now be more noticeable to our analyst that w_i must be determined by getting the ratio of the units of each column. From looking at the $ß_i y_{ji}$ values in Table 5, it is apparent that the unit for column 1 is the sum of all components and for columns 2 and 3, the components for Object 3. Thus, the analyst would compare the total of Box 1 (10 pounds), the ideal (largest) of Box 2 (8 pounds), and the worst (smallest) of Box 3 (6 pounds). In Table 5, we have chosen to show the resulting criteria weights in terms of the Object 3 component in Box 3. Using those ratios gives the correct relative overall preference (i.e. 3.333/2.833 = 1.176).

Table 5 – Muddled mode $ß_i y_{ji}$ and Composite Priorities

Criteria priorities=	1.667	1.333	1		
	Box 1	Box 2	Box 3	Simple Total	Weighted Total
Object 1	0.1	0.75	1.667	**2.5167**	2.833
Object 2	0.2	0.5	2.333	**3.0333**	3.333
Object 3	0.3	1	1	**2.3**	2.833
Object 4	0.4	0.25	1.667	**2.3167**	2.667
Total	**1**	**2.5**	**6.667**	**10.167**	

The derivation of criteria ratios in Table 5 is difficult because it is a mixture of three different methods: the distributive mode in column 1, the ideal mode in column 2 and the linking pins mode (Schoner et al, 1993) in column 3. The referent for the unit in columns 2 and 3 are relatively easy to visualize, since they are actual items in those boxes. For column 1, however, the unit is the summation of the 4 items in Box 1 and that unit is a value that is bigger than any of the individual items of which it is comprised. In this particular example, the concatenation of the four items in Box 1 (10 lbs) yields the correct criteria weights when compared against the item 3 in boxes 2 and 3. We caution that this outcome is a property of linearity in our sample problem that does not necessarily exist in other AHP problems.

To overcome possible non-linearity in unit sum scales, we suggest that the column total should not be used as the unit of comparison for deriving w_i. Instead, it is advisable to use a typical referent within the range of each criterion 's alternatives and then adjust to get the w_i value. For example, item 3 could be used as the referent for Box 1 along with item 3 of the other two boxes. Then, we would be comparing items weighing 3, 8 and 6 pounds from Boxes 1-3 respectively. The resulting ratios with item 3 of Box 3 as the unit would be .5, 1.333 and 1 for w_1, w_2 and w_3. However, we can see from Table 5 that item 3 of Box 1 is only .3 of the unit of column 1. Hence we would adjust that amount by .5/.3 to get 1.667 for w_3. In effect, the .5 value of item 3 used as the unit for column 1 is transformed into the distributive mode unit. Had we used item 2 across all three columns as the items for comparison, then the resulting ratios would have been .1, .2 and .7, which in turn are .2, .5 and 2.333 of the column units in Table 5. Consequently, the adjustments would be .1/.2 =.5 for w_1, .2/.5 = .4 for w_2 and .7/2.333 =.3 for w_3. Since these criteria weights are in the same ratio as 1.667, 1.333 and 1 in Table 5, they produce the same ratio of overall preferences.

3.0 Discussion of the Various Modes

In AHP, ratio scaled local priorities are established by evaluating the items under every node of the hierarchy. Here, we have specified y_{ji} as the ratio values so established and $ß_i y_{ji}$ as those priorities normalized in a specific manner. Although y_{ji} and $ß_i y_{ji}$ are in ratio form within a criterion, they are not commensurate across criteria. Accordingly, they cannot be added to get overall composite estimates of Vj. The Simple Total column in Tables 2 to 5 represents this. In all cases, ratios of simple totals do not produce the correct known ratios. In order to get correct f_j estimates, appropriate w_i values must be established and used.

As shown with the various modes, but particularly for the muddled mode, the appropriate w_i weights depend upon the manner in which the y_{ji} ratios have been normalized. For the distributive mode where the total criterion possessed by all the alternatives becomes the unit, the appropriate w_i values are determined by comparing those criterion totals. Alternatively, a typical alternative can be compared for each criterion and then adjusted to represent the distributive unit. For the ideal mode where the ideal alternative assumes unit value, appropriate w_i values are determined by comparing the ideal alternatives of each criterion. For the muddled mode where the $ß_i y_{ji}$ unit of each criterion is established in a different manner, the appropriate w_i values are determined by comparing the referent alternative(s) that form each unit.

When determining criteria weights for the distributive mode, an important distinction should be made. It is not the sum or total of the alternatives that is compared, but rather the degree of criterion possessed by all alternatives. This may appear to be a trifling point, but it is important. Since criteria weights are being ascertained, it must be the intensity of the criteria that are present that must be the focus.

The conclusion that must be drawn from these facts is that appropriate w_i values depend upon how $ß_i$ values have been established. A bottom up approach to establishing priorities is more conducive to illuminating this fact. Alternatively if a top down approach is used, the $ß_i$ values must be set in harmony with how previous w_i values were established. This point is important, because how w_i values are established is not well defined in the literature (Choo et al, 1999).

3.1 Changing $ß_i$ values

Conversion of the distributive mode to the ideal mode is a good example of the need to recognize the relationship between $ß_i$ and w_i values. Very often, the $ß_i y_j$ of the distributive mode are converted to the ideal mode by applying $ß_i = 1/\text{best } y_{ji}$ to the distributive mode priorities. This change in $ß_i$ values that converts Table 3 into Table 4 is a legitimate transformation since it maintains ratio relationships within criteria. The only difference is that the units of measure have changed.

But what is often done next is incorrect. The distributive mode w_i are then applied to the ideal mode $ß_i y_j$. When this happens, the ideal alternative is no longer the link for criteria comparisons, but it remains the link for composition. The result is incorrect ratios. For example, applying the criteria weights of Table 3 to the $ß_i y_j$ of Table 4 yields composite weighted totals of 4.607, 5.500, 4.464, and 4.357 for alternatives A_1 to A_4 respectively. This produces incorrect overall preferences ($5.5000/4.607 = 1.194 \neq .286/.243 = 1.176$). In order to get the correct values after changing $ß_i$ values, it is necessary to reassess and adjust w_i values.

Another important point to observe is that the ideal mode is put forward as a method that is immune to rank reversals. While this is true so long as the same ideal alternative continues to be normalized to unity, the ideal mode may suffer a more serious defect: its overall preference values will be wrong if the w_i are taken from another method.

3.2 Adding or deleting alternatives

Another important fact to note is that the addition or deletion of an alternative can change the unit of $ß_iy_j$ values if renormalization takes place after the change. This is particularly a problem for the distributive mode, since its local priorities and unit depend upon the set of alternatives being used. Additions or deletions change the set.

The effect of addition or deletion on the various modes is best illustrated by reference to the muddled mode. If we remove the components of object 4 from all boxes, then the $ß_iy_j$ values for columns 2 and 3 of Table 5 would remain unchanged. However, the distributive $ß_iy_j$ values for column 1 would have to be renormalized from .1, .2, .3 to .167, .333 and .5. They are still in the same ratio form, but the previous $ß_i$ has changed and the unit for the column now represents the total of three items.

Applying the criteria weights of Table 5 to the reduced $ß_iy_j$ of Table 5 yields the incorrect overall preferences. To capture the correct overall preferences, we would have to change w_1 of Table 5 to 1. This can be looked upon in the following way. Since item 4 in box 1 represented .4 of criterion 1, only .6 remains. Therefore, .6(1.667) = 1 for the new criterion weight. Using criteria weights of 1, 1.333 and 1 on the reduced matrix for Table 5 yields the correct overall priorities.

We note that renormalization upon removal or addition with the distributive mode requires the use of a new $ß_i$. As noted above, whenever $ß_i$ is changed, the w_i must be reassessed and revised. Failure to undertake that reassessment can lead to incorrect overall preference ratios. Some people on both side of the rank reversal debate fail to recognize that addition or deletion in the distributive mode is an alteration of $ß_i$ and a change in unit. As a result of this oversight, they fail to see that ratios change and commensurability is destroyed even though ranks may or may not have changed. Since AHP is ratio measurement, we contend that the rank reversal debate has been fought on the wrong grounds. Ordinal arguments have been used for a ratio problem. Had ratio stability been the goal, perhaps critics would have realized that rank reversal can be caused by changes in $ß_i$ that do not have a corresponding change in w_i.

3.3 Identifying units of measure

Since AHP is based upon ratio measurement, natural zero and a unit of measure identify its derived scales. But how can a person understand and identify the unit of measure of each scale so that derived scales can be transformed into another unit that is commensurate across criteria? We choose Table 5 for illustrative purposes, because its muddled magnitudes use different units of measure. There, different units are used for each column.

The units of measure in Table 5 for Box 2 and 3 are fairly explicit – they are the best item in Box 2 and the worst item in Box 3. In both cases, those items belong to Object 3. When deriving the local priorities for the items in those boxes, item 3 is the denominator or base to which all other items are compared. The priorities off all other items are relative to item 3.

The unit of measure for the items in Box 1 is more difficult to ascertain. In the distributive mode, the total rather than the local preference of any one alternative takes the value of unity. This implies that the priorities of all the items are relative to all of themselves and that the total criterion possessed by all relevant alternatives should be the unit to compare when deriving criteria weights. The problem with this approach is that it is more difficult to think of the totality of a criterion than the amount of criterion possessed by a single alternative.

We suggest that a way around this cognitive problem is to take a specific alternative from each criterion and pair compare the criteria they possess. Then, these linking priorities for criteria can be scaled upward to reflect the unit sum totality of all local priorities. This process makes the criteria comparisons more similar to the ideal and linking pin mode where the unit alternative is more visible. We suggest that this process is better able to handle non-linear relationships between derived scales and their individual components.

4.0 Conclusion

Our use of muddled magnitudes is for illustrative purposes only. In practice, we would not recommend such muddling, because it increases the cognitive load when making criteria comparisons. Nevertheless, the illustration is useful, because the muddled magnitudes, placed side by side, illustrate that different normalizations of derived scales lead to different units of measure. Too frequently, the analyst is unaware that different units exist and that renormalizations produce new units. Failure to recognize this fact can lead to faulty solutions.

In (1), f_j, the overall preference, can be looked upon as the sum of partial $w_i ß_i y_{ji}$ values. In order for summation of those partial values to yield a ratio answer, their individual partial values, before addition, must be in commensurate units.

A problem arises because y_{ji} and $ß_i y_{ji}$, although ratio, are not in commensurate units. The function of criteria weights is to rescale $ß_i y_{ji}$ values into commensurate partial values so that they can be added. Not just any set of w_i achieves the commensurability.

Recognition of the unit of the derived scales assists in generating valid criteria weights. Comparisons across criteria to get w_i should be done in reference to the unit of the derived scale. This produces w_i values for the units that are ratio measures across criteria. Through linkages to the unit items during hierarchic weighting, the relative criteria importance is transferred to all alternatives under that criterion. Now, partial values are commensurate and summation can take place.

Important in this procedure is the fact that the normalization process of the derived scale determines its unit of measure. Thus, $ß_i$ determines the unit. But since knowledge of the unit is necessary to determine w_i, the two measures are intractably linked. Any change in $ß_i$ requires a corresponding reassessment of w_i in order to generate correct overall preferences. Such a reassessment is required if alternatives are added or deleted with the distributive mode or if the form of normalization is changed. Knowledge of this will lead to composite priorities that measure valid overall preferences.

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