THE RELIABILITY OF DATA

IN PAIRWISE COMPARISON MATRICES

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ABSTRACT

This article analyzes the problem of reliability of data collected in the pairwise comparison matrices used in the Analytic Hierarchy Process for prioritization of alternatives. The hierarchy obtained from the inappropriate input data may be faulty. Basing on the consistency index defined by Saaty in 1977 we introduce a method which allows to search for the least reliable entries of a pairwise comparison matrix and replace them with their estimations.

Keywords: decision making, pairwise comparisons, inconsistency.

1. Introduction

The method of pairwise comparisons is widely applied in the decision making process. The inconsistency of data may significantly affect the final result. Many authors, starting from Saaty (1977), introduced various consistency indices which allow to judge how far a pairwise comparison matrix (PC matrix) is from consistency.

All these measures, however, do not indicate the source of inconsistency. Reasons for that can be different, depending on how we accumulate data. In most cases, pairwise comparisons are performed by experts. However, some of their judgments may be wrong or subjective. Sometimes, the entries of a PC matrix come from seemingly objective tests, like the results of sport's games. Even so, a number of them may also be misleading and not reflect a real strength of a sportsman or a team, as they may result from worse disposition of the day or other exceptional circumstances.

That is why it is important to find a method of eliminating such incorrect data. In some sport's disciplines (for instance figure skating or ski jumping) where the overall result depends on judges' evaluation, the extreme rates are not taken into consideration. The main idea for this study was to measure how far the elements of a PC matrix are from their consistent approximation obtained by a principle eigenvector method introduced by Saaty (1977).

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2. Literature Review

The pairwise comparison method has become very popular for multi criteria decision makers due to Saaty (1977), who introduced the Analytic Hierarchy Process (AHP) for finding priorities in hierarchical structures.

For a square matrix A with positive entries a_{ij}, satisfying the *reciprocity condition*

$$a_{ij} \cdot a_{ji} = 1,$$

But not satisfying the *consistency condition*

$$a_{ij} \cdot a_{jk} \cdot a_{ki} = 1,$$

and representing pairwise comparisons of *n* alternatives according to one of several criteria he suggested the prioritization procedure. The first step is to find the positive *principal eigenvalue* λ_{max} and its corresponding *eigenvector* $v=(v_1,\dots,v_n)$ with positive coordinates, satisfying

$$\mathrm{A}\, v = \lambda_{max}\, v.$$

Vector *v* becomes a priority vector and induces a perfectly consistent PC matrix S with the entries $s_{ij} = \frac{v_i}{v_j}$.

In the same paper Saaty defined the consistency index

$$CI = \frac{\lambda_{max} - n}{n-1},$$

which is equal to

$$CI = \frac{1}{n(n-1)} \sum_{i \neq j} \left(a_{ij} s_{ji} - 1 \right),$$

as it was shown by Aguarón & Moreno-Jiménez (2003).

The idea of the PC matrix inconsistency reduction by correction of the least reliable entries was given by Koczkodaj & Szybowski (2016).

3. Objective

The main objective of this study is to introduce an algorithm of improvement of the input data for the prioritization of alternatives by means of pairwise comparisons.

4. The Algorithm of the Input Data Improvement

Let $\mathrm{A} = [a_{ij}]$ be an inconsistent PC matrix. Fix a positive threshold ε.

The algorithm of the input data improvement is quite simple:
ALGORITHM (A: matrix)

```
begin
    repeat
    begin
        v:=PRINCIPLE_EIGENVECTOR(A);
        CI:=0; MAX:=0;
        for i:=1 to n-1
            for j:=i+1 to n
                begin
                    s_ij := v_i/v_j; s_ji := v_j/v_i;
                    AUX:= a_ij s_ji + a_ji s_ij - 2;
                    CI:= CI+AUX;
                    if AUX>MAX then
                    begin
                        MAX:=AUX; p:=i; q:=j;
                    end
                end
        CI:=CI/(n^2 - n);
        a_pq := s_pq; a_qp := s_qp;
    end
    until CI > ε
end
```

In each step of this algorithm we search for the most unreliable entry in the PC matrix and we replace it with its estimation obtained from ratios of the adequate principle eigenvector coordinates. The algorithm stops when the consistency index of the matrix becomes smaller then the assumed threshold.

5. Numerical Example

Let us investigate the algorithm started for a PC matrix introduced in one of the examples in Saaty (1977):

$$A = \begin{bmatrix} 1.000 & 3.000 & 7.000 & 9.000 \\ 0.333 & 1.000 & 6.000 & 7.000 \\ 0.143 & 0.167 & 1.000 & 3.000 \\ 0.111 & 0.143 & 0.333 & 1.000 \end{bmatrix}.$$

Let us fix the threshold ε=0.05.

For this matrix $CI(A) = 0.07$, so we continue with the algorithm. The principal eigenvector is equal to $v = (0.58, 0.30,\ 0.08, 0.04)$, and its consistent approximation is

$$S = \begin{bmatrix} 1.000 & 1.933 & 7.250 & 14.50 \\ 0.517 & 1.000 & 3.750 & 7.500 \\ 0.138 & 0.267 & 1.000 & 2.000 \\ 0.069 & 0.133 & 0.500 & 1.000 \end{bmatrix}.$$

The highest value 0.2313 of the expression $a_{ij}s_{ji} + a_{ji}s_{ij} - 2$ is obtained for $i = 1$ and $j = 3$, thus the matrix after the first step will be

$$A = \begin{bmatrix} 1.000 & 3.000 & 7.000 & 14.50 \\ 0.333 & 1.000 & 6.000 & 7.000 \\ 0.143 & 0.167 & 1.000 & 3.000 \\ 0.069 & 0.143 & 0.333 & 1.000 \end{bmatrix},$$

Having a reduced consistency index $CI(A) = 0.04$, which is smaller than 0.05, so the algorithm stops.

6. Limitations

The above algorithm has been invented for the improvement of a few PC matrix entries which could falsify the hierarchy of alternatives. However, its application in order to improve all or most of matrix elements seems to be pointless. Our main goal is always to obtain a priority vector, which can be done immediately, for example, by means of the AHP. If we know that most of input data is unreliable, we cannot expect that the output data will be satisfactory.

Another problem is that the algorithm does not guarantee the preservation of the scale introduced by Saaty. Obviously, it does not interfere with the application of AHP, but traditionalists may not like it.

7. Conclusions

We have proposed the algorithm of the pairwise comparison data improvement, which is fast and simple. What is left to do it is to formally prove its correctness, i.e. that the consistency indices of the matrices obtained in the subsequent steps tend to zero. Secondly, it should be implemented and carefully tested.

The least reliable PC matrix entries elimination also seems to be a perfect idea for the choice of the best matrix generators introduced by Koczkodaj & Szybowski (2015).

8. Key References

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