Judgment Scales of the Analytical Hierarchy Process – The Balanced Scale

ABSTRACT

One topic under discussion of the analytic hierarchy process is the use of different scales in order to translate judgments into ratios. The author shows that the so-called balanced scale has a uniform weight distribution for two decision criteria only. If it is applied to decision problems with more than two criteria, weights are no longer balanced, and priorities are underweighted. A generalization of the balanced scale is proposed, which takes into account the number of decision criteria. It is shown that the generalized balanced scale yields equally dispersed local weights for any number of decision criteria.

Keywords: Analytic Hierarchy Process, Judgment Scales, Balanced Scale, Generalized Balanced Scale, Pairwise Comparisons.

1. Introduction

Despite all academic discussions, the analytic hierarchy process (AHP) remains one of the most popular multi-criteria decision making methods (MCDM). Originally proposed by Saaty (1980), over the last decades several modifications and improvement have been proposed. An overview was given, for example, by Ishizaka &Labib (2011). One of the topics being under discussion for a long time is the fundamental AHP scale. Saaty and Vargas (2012) describe ratio scales, proportionality and normalized ratio scales as one of the seven pillars of the Analytic Hierarchy Process. The fundamental AHP scale of absolute numbers is derived from the psychophysical law of Weber–Fechner and uses absolute numbers 1, 2, 3 … 9 or its verbal equivalents (Saaty, 2008).

Paired comparisons are made by identifying the less dominant of two elements and using it as the unit of measurement. One then determines, how many times more the dominant member of the pair is than this unit. The reciprocal value is used for the comparison of the less dominant element with the more dominant one.

Theoretically there is no reason to be restricted to these numbers and verbal gradation, and several other numerical scales have been proposed. Salo & Hämäläinen (1997) pointed out that the integers from 1 to 9 yield local weights, which are not equally dispersed. For example, a judgment change from $x = 1$ to 2 yields to a weight change of 17%, whereas a judgment change from $x = 8$ to 9 results in a weight change of only 1.1%; a factor of 15-times lower. There is a lack of sensitivity, when comparing elements close to each other.

They state that for a given set of priority vectors w_{AHP} the corresponding ratios r can be computed from the relationship

$$r = \frac{w_{AHP}}{1-w_{AHP}} \qquad (1a)$$

or
$$w_{\text{AHP}} = \frac{r}{r+1} \tag{1b}$$

Using a scale
$$c = \frac{w_{\text{bal}}(x)}{1-w_{\text{bal}}(x)} \tag{2a}$$

with evenly dispersed weights
$$w_{\text{bal}} = 0.45 + 0.05\,x \tag{2b}$$

and judgments $x = 1 \ldots 9$ yield equally distributed weights from 50% to 90%.

The balanced scale can be written as
$$c = \frac{9+x}{11-x} \tag{2c}$$

c (resp. 1/*c*) are the entry values in the decision matrix, and *x* the pairwise comparison judgment on the fundamental 1 to 9 judgment scale.

2. Analysis and Discussion

The Generalized Balanced Scale

In fact, eq. 1a or its inverse eq. 1b are a special case for *one pairwise comparison of two criteria*. If we take into account the complete *n* x *n* decision matrix for *n* criteria, the resulting weights for a criterion, judged *x*-times more important than all others, can be calculated as (see appendix, eq. a5):
$$r = \frac{w_{AHP}}{1-w_{AHP}}(\text{n}-1) \tag{3a}$$

$$w_{\text{AHP}} = \frac{r}{r+n-1} \tag{3b}$$

Eq. 3b simplifies to eq. 1b for *n*=2.

We now use eq. 3a to formulate the more general case of the balanced scale for *n* criteria and a judgment *x* with *x* from 1 to *M*, resulting in evenly dispersed weights:
$$c = \frac{w_{\text{bal}}}{1-w_{\text{bal}}}(\text{n}-1) \tag{4a}$$

with evenly dispersed weights
$$w_{\text{bal}}(x) = w_{\text{eq}} + \left[\frac{w_{\max}-w_{\text{eq}}}{M-1}\right](x-1) \tag{4b}$$

using
$$w_{\text{eq}} = \frac{1}{n} \tag{4c}$$

and
$$w_{\max} = \frac{M}{n+M-1} \tag{4d}$$

we get the generalized balanced scale as
$$w_{\text{bal}} = \frac{1}{n} + \left[\frac{\frac{M}{n+M-1}-\frac{1}{n}}{M-1}\right](x-1) \tag{5a}$$

Setting *M* = 9 the generalized balanced scale can be written as
$$c = \frac{9+(n-1)x}{9+n-x} \tag{5b}$$

We see that eq. 5b with *n*=2 represents the classical balanced scale as given in eq. 2c. We call eq. 5b the *generalized balanced scale* or balanced-n (bal-n) scale.

In order to compare the local weights as a function of the judgments *x* with the number of criteria *n* as parameter, we use eq. 2c in eq. 3b to reflect the actual weights of the classical balanced scale for more than two criteria ($n > 2$).

Balanced scale $$w_{\text{AHP}} = \frac{x+9}{(2-n)x+11n-2} \quad (6a)$$

AHP fundamental scale $$w_{\text{AHP}} = \frac{x}{x+n-1} \quad (6b)$$

Generalized balanced scale $$w_{\text{AHP}} = \frac{9+(n-1)x}{n(n+8)} \quad (6c)$$

Fig. 1 visualizes the three functions for $n = 7$ criteria. It can be seen that a single judgement "5 – strong more important" yields to a weight of 45% on the AHP scale, 37% on the generalized balanced scale and 28% on the balanced scale. Equations 6a, b and c show that for all *n*, criteria are underweighted using the classical balanced scale and over weighted using the fundamental AHP scale, except for $n = 2$, where the classical balanced scale is identical with the generalized balanced scale and yields evenly distributed weights.

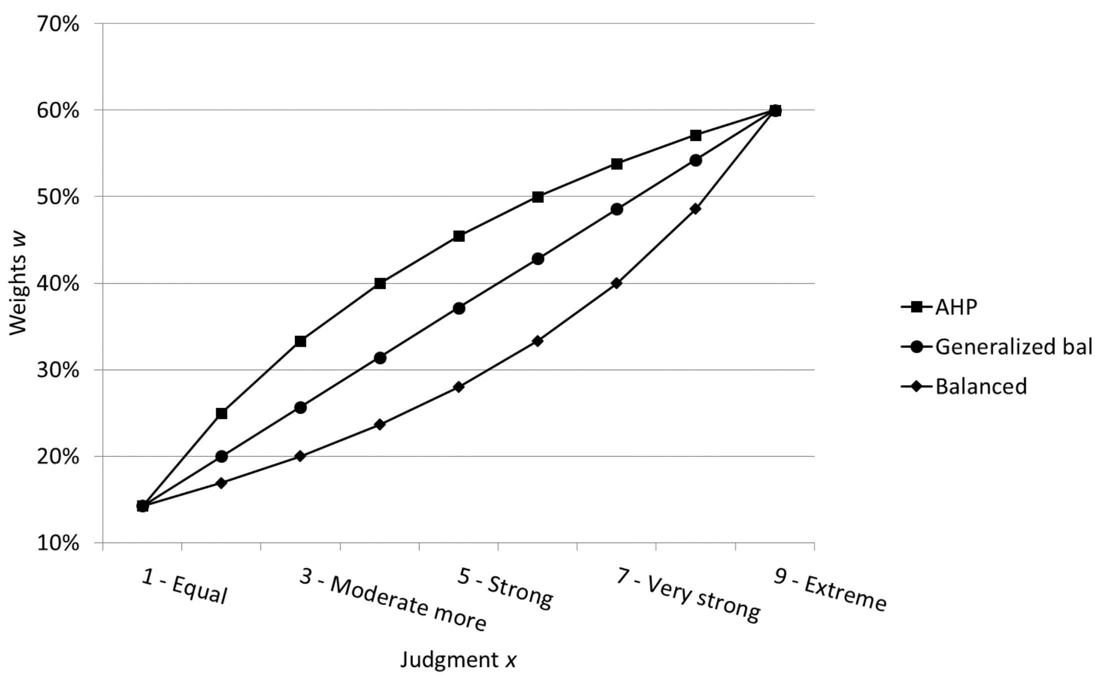

Figure 1. Visualization of eq. 6a, b and c: local weights as function of judgment x for the fundamental AHP scale, the balanced scale and the generalized balanced scale for $n = 7$ decision criteria.

Weight dispersion

In order to compare the weight dispersion of the fundamental, the balanced and generalized balanced scale, we can calculate the derivative of eq. 3b replacing *r* with the scale function $c(x)$. A balanced scale should yield a constant value independent of the judgment *x*. For the different scales the derivative of eq. 3b with 6a, b and c gives us

Fundamental AHP scale $$\frac{\mathrm{dw_{AHP}}}{\mathrm{dx}} = \frac{n-1}{(x+n-1)^2} \quad (8a)$$

Balanced scale $$\frac{\mathrm{dw_{AHP}}}{\mathrm{dx}} = \frac{20(n-1)}{((n-2)x-11n+2)^2} \quad (8b)$$

Generalized balanced scale $$\frac{\mathrm{dw_{AHP}}}{\mathrm{dx}} = \frac{n-1}{n(n+8)} \quad (8c)$$

Only for the generalized balanced scale the derivative does not depend on the judgment *x*; weights are equally distributed over the judgment range. For $n = 2$ eq. 8b is identical with eq. 8c; each integer step on the 1 to 9 scale increases the local weight by 5%.

3. Conclusions

The so-called balanced scale has to be generalized and has to take into account the number of criteria in order to be applied for more than two criteria. Otherwise local priorities will not be balanced and will be underweighted compared to the generalized balanced scale and the fundamental AHP scale. The generalized balanced scale takes into account the number of decision criteria and improves weight dispersion compared to balanced scale and the original AHP scale.

4. Key References

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5. Appendix: AHP weights as a function of judgments

Let *DM* be a *n* x *n* decision matrix, where the first criterion is *x*-times more important than all others. Then the first matrix element is “1”, and the rest of the first matrix row is filled with (n-1)-times *x*. The first matrix column is filled with (n-1)-times 1/x.

$$DM = \begin{pmatrix} \mathbf{1} & x & x \\ 1/x & \mathbf{1} & 1 \\ 1/x & 1 & \mathbf{1} \end{pmatrix} \qquad \text{(a1)}$$

To calculate the priorities, we use the Row Geometric Mean Method (RGGM, Crawford 1985), as the decision matrix is consistent and the result will be the same as for the right eigenvector.

$$\text{RGGM} \rightarrow \begin{pmatrix} (x^{n-1})^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \end{pmatrix} \qquad \text{(a2)}$$

The resulting weights (priorities) for the first criterion is the normalized geometric mean of the first row.

$$w_{\text{AHP}} = \frac{(x^{n-1})^{\frac{1}{n}}}{(x^{n-1})^{\frac{1}{n}} + (n-1)(x^{-1})^{\frac{1}{n}}} \qquad \text{(a3)}$$

With some rearrangement

$$w_{\text{AHP}} = \frac{1}{1+\frac{(n-1)(x^{-1})^{\frac{1}{n}}}{(x^{n-1})^{\frac{1}{n}}}} = \frac{1}{1+\frac{(n-1)x^{-\frac{1}{n}}}{x \cdot x^{-\frac{1}{n}}}} = \frac{1}{1+\frac{(n-1)}{x}} \qquad \text{(a4)}$$

we get the simple relation

$$w_{\text{AHP}} = \frac{x}{x+n-1} \qquad \text{(a5)}$$

□