AHP PRIORITIES AND MARKOV-CHAPMAN-KOLMOGOROV STEADY-STATES PROBABILITIES

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ABSTRACT

An AHP matrix of the quotients of the pair comparison priorities can be transformed to a matrix of shares of the preferences. The transformed matrix can be used in Markov stochastic modeling via the Chapman-Kolmogorov system of equations for the discrete states. It yields a general solution and the steady-state probabilities. The priority vector can be interpreted as the eventual probabilities to belong to the discrete states corresponded to the compared items. The results of stochastic modeling correspond to robust estimations of priority vectors.

Keywords: AHP, Markov stochastic modeling, Chapman-Kolmogorov equations.

1. Introduction

The work describes relations between AHP and Markov stochastic modeling via Chapman-Kolmogorov equations for discrete states. The suggested approach is based on transformation of a Saaty matrix of the priority quotients to a matrix of the preference shares. This share matrix is used in constructing Chapman-Kolmogorov system of differential equations and solving it for the dynamic and eventually reached steady-state probabilities which define preferences among the compared items. The solution corresponds to the eigenproblem for obtaining robust priority vectors.

2. Literature Review

The Analytic Hierarchy and Analytic Network Processes (Saaty, 1980, 1996) are ones of the most widely known and applied methods of multiple criteria decision making. We consider relations of the AHP with some other techniques of pairwise comparison data, including the Thurstone scaling, Bradley-Terry-Luce, and Markov stochastic model presented as Chapman-Kolmogorov equations for discrete states, and some results are presented in (Lipovetsky & Conklin, 2002, 2003; Lipovetsky, 2005).

3. Hypotheses/Objectives

In this study we consider a possibility to use a data of Saaty pairwise comparison matrix in the stochastic model of transition of preferences among the states corresponding to the compared alternatives. Finding the steady-state preferences opens a possibility of interpretation of the AHP priorities as the probabilities of choices among the items under considerations.

4. Research Design/Methodology

A theoretical Saaty matrix of pair comparisons for *n* items defines each *ij*-th element as a ratio of unknown priorities w_i and w_j:

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ ----- & ------- & & ---- \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix} \quad (1)$$

Multiplying matrix (1) by the vector $w = (w_1, w_2, ..., w_n)'$ we get identical relation

$$Ww = nw \,, \quad (2)$$

Elicited from a judge, an empirical pair comparison matrix of priority ratios is

$$A = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots & 1 \end{pmatrix}. \quad (3)$$

It is a Saaty matrix with transposed-reciprocal elements

$$a_{ij} = a_{ji}^{-1} \,. \quad (4)$$

Similarly to (2), priorities in the AHP are estimated by the eigenproblem for a matrix (3):

$$A\alpha = \lambda\alpha \,, \quad (5)$$

where the maximum eigenvalue corresponds to the term *n* in (2), and the principal eigenvector α estimates the vector of priorities *w*.

Let us introduce a theoretical matrix of shares

$$U = \begin{pmatrix} w_1/(w_1+w_1) & w_1/(w_1+w_2) & \dots & w_1/(w_1+w_n) \\ ----- & ------- & & ---- \\ w_n/(w_n+w_1) & w_n/(w_n+w_2) & \dots & w_n/(w_n+w_n) \end{pmatrix}, \quad (6)$$

Each element u_{ij} is defined as *i*-th priority in the sum of *i*-th and *j*-th theoretical priorities:

$$u_{ij} = \frac{w_i}{w_i + w_j} = \frac{w_i / w_j}{1 + w_i / w_j} \,. \quad (7)$$

To estimate priority vector by the matrix (6) we write identical equalities:

$$\begin{cases} \frac{w_1}{w_1+w_1}(w_1+w_1)+\frac{w_1}{w_1+w_2}(w_1+w_2)+...+\frac{w_1}{w_1+w_n}(w_1+w_n)=nw_1 \\ - - - - - - - - - - - - - - - - - - - \\ \frac{w_n}{w_n+w_1}(w_n+w_1)+\frac{w_n}{w_n+w_2}(w_n+w_2)+...+\frac{w_n}{w_n+w_n}(w_n+w_n)=nw_n \end{cases} . \quad (8)$$

Then using notation (7) we present the system (8) as:

$$\begin{cases} (u_{11}+\sum_{j=1}^{n} u_{1j})w_1+u_{12}w_2+...+u_{1n}w_n=nw_1 \\ - - - - - - - - - - - \\ u_{n1}w_1+u_{n2}w_2+...+(u_{nn}+\sum_{j=1}^{n} u_{nj})w_n=nw_n \end{cases} \quad (9)$$

In the matrix form the system (9) is:

$$\left(U+diag(Ue)\right)w=nw, \quad (10)$$

where U is the matrix (6), e denotes a uniform vector of n-th order, and $diag(Ue)$ is a diagonal matrix of totals in each row of matrix U. Relations (8)-(10) for the theoretical matrix of shares (6) are derived similarly to the problem (2) for the matrix (1).

In classical AHP, pair ratios w_i/w_j (1) are estimated by elicited values a_{ij} (3). Using a_{ij} in (7) we obtain empirical estimates b_{ij} of the pairs' shares:

$$b_{ij}=\frac{a_{ij}}{1+a_{ij}}$$

(11)

This transformation of the elements of a matrix A (3) yields a pairwise share matrix B with elements (11). The elements of such a matrix (11) are positive, less than one, and have a property of symmetry:

$$b_{ij}+b_{ji}=1 . \quad (12)$$

It means that the transposed elements b_{ij} and b_{ji} are equidistant from the diagonal elements b_{ii}=0.5, so $b_{ij}-b_{ii}=b_{ii}-b_{ji}$. Elements of a Saaty matrix (3) with large or small values are transformed in (11) to the values closer to one or zero, respectively.

In the AHP, for empirical Saaty matrix A (3) we have eigenproblem (5) in place of theoretical relations (2). By the same pattern, using empirical matrix B (11) in place of theoretical matrix U, we represent the system (10) as an eigenproblem

$$\left(B+diag(B\,e)\right)\alpha=\lambda\alpha, \quad (13)$$

where α as a vector of priority. Multiplying the matrix in (13) by the uniform vector and using (12) shows that this matrix has a property of the total in its each column equals *n*:

$$\left(B + diag(B\,e)\right)' e = B'e + Be = n\,e\ , \qquad (14)$$

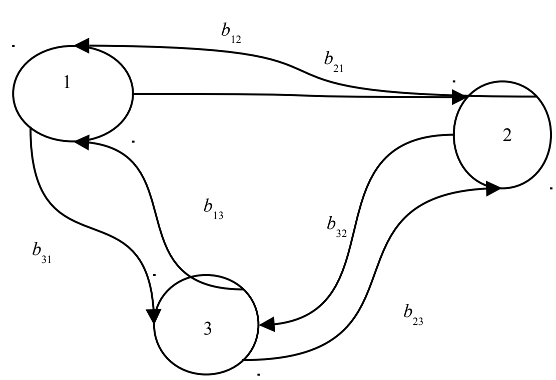
where prime denotes transposition. Dividing both sides of equations (13) by the term *n*, we obtain eigenproblem of a positive matrix with totals in the columns equal to one, which is an eigenproblem of the transposed stochastic matrix. Such a matrix has the maximum eigenvalue equals one. Due to the Perron-Frobenius theory for a positive matrix, its main eigenvector always exists, is a unique one, and has all positive elements. Thus, the maximum eigenvalue in (13) equals *n*, and a solution for the main eigenvector exists and is unique, that ensures in the desired properties of the priority vector.

5. Data/Model Analysis

Eigenproblem (13) has a matrix a transposed stochastic kind that relates it to matrices known in Markov modeling. Consider a discrete state and continuous time Markov model presented via Chapman-Kolmogorov differential equations describing a stochastic process of transitions among the states. This model is based on properties of a finite set of the elements (alternatives compared within a criterion) that are tied by the constant transition probabilities of each alternative's prevalence over the others. The prevalence of one item over another one in the AHP corresponds to probability of the former item to be preferred over the latter one in the eliciting process. The Chapman-Kolmogorov equations express change in probability to be found in any of *n* states as a linear combination of these probabilities with the coefficients of the transition intensities.

Taking a pair of the elements b_{ij} and b_{ji} of the share matrix (11) we notice that each element can be interpreted in terms of probability to prefer one of the items over another one, due to the meaning of the theoretical shares (7). The preference of an *i*-th item over a *j*-th item corresponds to transition between them with intensity b_{ij}. The share matrix *B* can be presented as a connected oriented graph with *n* nodes of states/alternatives and two edges between each of pair of nodes, one going to state *i* from state *j* corresponds to transition intensity b_{ij} and the other going from state *i* to state *j* corresponds to transition intensity b_{ji}. An example of such a network is shown in Figure 1.

Figure 1. AHP Network of the Transition Shares



The system of Chapman-Kolmogorov equations can be presented as following:

$$\begin{cases} \dfrac{dp_1}{dt} = (b_{12}p_2 + \ldots + b_{1n}p_n) - (b_{21}p_1 + \ldots + b_{n1}p_1) \\ - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ \dfrac{dp_n}{dt} = (b_{n1}p_1 + \ldots + b_{n,n-1}p_{n-1}) - (b_{1n}p_n + \ldots + b_{n-1,n}p_n) \end{cases} \quad (15)$$

where p_i denotes probability to belong to an i-th state, and coefficients b_{ij} are the values (11). Items with positive signs at the right-hand side (15) define influx to each state from all the others, and those with negative signs define departure from a state to all the other states. If canceling items $0.5p_i$ are added to both positive and negative inputs in each i-th equation (15), this system can be represented in the matrix form:

$$\dot{p} = \left(B - diag(B'e)\right) p \,, \quad (16)$$

where p is a vector of the probabilities p_i for all the states, $\dot{p}$ denotes the vector of their derivatives, B is the same matrix with elements (11), B' is its transposition, and e is the identity vector. Using property (14) that the sum of totals in i-th column and row of the matrix B equals n, we can rewrite (16) as:

$$\dot{p} = \left(B + diag(B\,e) - nI\right) p \,, \quad (17)$$

where I denotes the identity matrix of n-th order.

Considering solution of the Chapman-Kolmogorov equations (17) for the steady-state probabilities when the process is stabilized, we put the derivatives in the left-hand side equal zero, and (17) reduces to:

$$\left(B + diag(B\,e)\right) p = n\,p \,. \quad (18)$$

But (18) is nothing else but the same eigenproblem (13) with the largest eigenvalue $\lambda = n$ and a unique positive main eigenvector, as it was discussed in relation to equality (14). So the results of the AHP priority evaluation (13) can be interpreted from the point of view of the stochastic process steady-state solution (18) as follows: the AHP priority vector corresponds to the eventual probabilities to belong to the discrete states, or alternatives, and these probabilities define the preferences among the compared items.

6. Limitations

The described approach should be further proved by numerical modeling, and we are going to perform it and compare the results for classical and transformed AHP matrices and priority vectors.

7. Conclusions

Transformation of the pairwise ratio AHP matrices to the pairwise share matrices, and solving the corresponding eigenproblem is considered. This approach can be obtained in Chapman-Kolmogorov modeling of transitions among the discrete states of the alternatives. Coincidence of these results for AHP priority evaluation and of the stochastic steady-state solution suggests a useful interpretation: the AHP priorities have a meaning of the eventual probabilities to belong to the discrete states of the compared items. The results of priority evaluation in terms of the choice probabilities can be used in theoretical modeling and practical applications for various multiple criteria decision making problems.

8. Key References

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