Consistency of expert-based preference matrices

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1. Cyclic inconsistency of a preference matrix

In AHP approach to multi-criteria decision problem, the relative importance of alternatives is computed from preference matrices, which come from experience and can possibly be inconsistent.

An algorithm for computing a consistent approximation of a given preference matrix by digraph method is described in this paper. We start with an analysis of the inconsistency of a given preference matrix. The first type of inconsistency is caused by so-called inconsistency cycles. The inconsistency of this type is removed by computing the strongly connected components in the associated digraph and a small modification. If the modified matrix is cyclic consistent, i.e. it contains no inconsistent cycles, or if some of the entries of the matrix are missing, then a consistent approximation is computed. The computational complexity of the algorithm is $O(n^2)$.

Preference matrix $A$ is called cyclic inconsistent, if there is a cycle

$$i_1, i_2, \ldots, i_r, i_{r+1} = i_1$$

of length $r \geq 2$ of indices in $N$ (called: inconsistent cycle in $A$) such that the inequalities

$$a(i_ki_{k+1}) \geq 1 \quad \text{for every} \quad k = 1, 2, \ldots, r$$

(1)

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hold, and at least one of the inequalities is strict. Matrix $A$ is cyclic consistent, if every cycle in $A$ is consistent, i.e. $A$ contains no inconsistent cycles. Examples of inconsistent cycles of length $r = 3$ are shown in Figure 1.

![Figure 1: Possible types of cyclic inconsistency with $r = 3$](image)

**Theorem 1.1.** If a complete preference matrix $A$ contains an inconsistent cycle of length $r > 3$, then $A$ also contains an inconsistent cycle of length 3.

**Corollary 1.2.** The cyclic consistency of a complete preference matrix can be recognized in time $O(n^3)$, by verifying all index cycles of length 3 for the inconsistency.

Theorem 1.1 is illustrated by Figure 2 below.

![Figure 2: Cyclic inconsistency with $r = 4$](image)

The preference digraph $\mathcal{D} = (V(A), E(A))$ of a given preference matrix $A$ is defined as follows

\[
V(\mathcal{D}) = \{1, 2, \ldots, n\} \\
E(\mathcal{D}) = \{(i, j); a_{ij} \geq 1\} \\
E^+(\mathcal{D}) = \{(i, j); a_{ij} > 1\}
\]

The edges in $E^+(\mathcal{D})$ are called strong preference edges.
Theorem 1.3. Preference matrix $A$ is cyclic consistent if and only if every cycle $C$ in $\mathcal{D}(A)$ contains no strong preference edges, i.e. $C \cap E^+(\mathcal{D}) = \emptyset$.

Theorem 1.4. Preference matrix $A$ is cyclic consistent if and only if every strongly connected component $K$ in $\mathcal{D}(A)$ contains no strong preference edges, i.e. $(K \times K) \cap E^+(\mathcal{D}) = \emptyset$.

A possible treating method (3-cycle method, for short) is based on Theorem 1.1.

3-cycle method
1. find all inconsistent cycles of length 3 in $A$
2. change all preferences in the cycles to 1
3. repeat 1-2 until there is no inconsistent cycle of length 3 in $A$

The repetition of steps 1 and 2 in the 3-cycle method is necessary, because new inconsistent cycles can be created by treating the cyclic inconsistency by cycles of length 3 (see Figure 3).

The above disadvantage is not present at another method for treating the inconsistency (SCC method, for short) which works with strongly connected components in the preference digraph $\mathcal{D}(A)$. The method is based on Theorem 1.4.

SCC method
1. find all strongly connected components in digraf $\mathcal{D}(A)$
2. change all preferences within the strongly connected components to 1

Figure 3: Creating new inconsistent cycles of length $r = 3$
Theorem 1.5. If matrix $A'$ is created from preference matrix $A$ by SCC method, i.e.
$$a'_{ij} = \begin{cases} 1 & \text{for } i, j \in \mathcal{K} \text{ in every strongly connected component } \mathcal{K} \text{ of } \mathcal{D}(A) \\ a_{ij} & \text{otherwise} \end{cases},$$
then $A'$ is reciprocal and cyclic consistent.

The cyclic consistent matrix $A'$ computed by the SCC algorithm is called the cyclic consistent approximation of $A$.

2. Computing consistent preferences

Preference order $\mathcal{P}(A)$ induced by $A$ is defined as follows: if inequalities $a(i_ki_{k+1}) \geq 1$ with $k = 1, 2, \ldots, r - 1$ hold for some sequence $i = i_1, i_2, \ldots, i_r = j$, then $(i, j) \in \mathcal{P}(A)$.

Theorem 2.1. If a preference matrix $A$ is cyclic consistent, then $\mathcal{P}(A)$ is a uniquely determined linear order of alternatives (up to permutations of equivalent preferences).

Theorem 2.2. If $A$ is a preference matrix and $A'$ is its cyclic consistent approximation, then the linear order $\mathcal{P}(A')$ of alternatives is equal to the order of strictly connected components in preference digraph $\mathcal{D}(A)$.

Algorithm ConsistApprox

1. Input: preference matrix $A$
2. compute preference digraph $\mathcal{D}(A)$
3. compute cyclic consistent matrix $A'$ by SCC method
4. compute linear order $\mathcal{P}(A')$ induced by the order of strongly connected components in digraph $\mathcal{D}(A)$
5. for any component $\mathcal{K}$ and its successor $\mathcal{L}$ substitute values $a'_{ij}$ with $i \in \mathcal{K}$, $j \in \mathcal{L}$ by their common geometric mean $\tilde{a}_{ij}$
6. extend the ‘overdiagonal block’ values in matrix $\tilde{A}$ using the reciprocity and consistency condition
7. Output: reciprocal and consistent approximation matrix $\tilde{A}$

Theorem 2.3. Algorithm ConsistApprox works correctly and for every $n \times n$ preference matrix $A$ the algorithm computes a consistent approximation $\tilde{A}$ in $O(n^2)$ time.
Remark 2.4. Algorithm ConsistApprox can easily be modified also for the case of missing values in the input preference matrix $A$. In the modification, the missing input values create no edges in the preference digraph $D(A)$.

Theorem 2.5. If $A$ is a preference matrix with missing values and if the order of strongly connected components in the preference digraph $D(A)$ is linear, then algorithm ConsistApprox with input matrix $A$ works correctly, and the algorithm computes a consistent approximation $	ilde{A}$ in $O(n^2)$ time.

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